Bottom-Up Reconstruction of Viable GW170817 Compatible Einstein-Gauss-Bonnet Theories

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Overview

Lovelock Gravity

Lovelock's theory of gravity is one of the scalar - tensor theories of gravity.

These theories are unique in requiring no extra fundamental fields beyond those that go into GR, while maintaining the property that the field equations of the theory can be written with no higher than second derivatives of the metric.

Lovelock asked which set of rank-2 tensors $A^{\mu\nu}$ could satisfy the following two conditions:

i) $\nabla_\nu A^{\mu\nu} = 0$

ii) $A^{\mu\nu} = A^{\nu\mu}$

He found that there was a considerably broader class of solutions to the problem, each of which could serve as a suitable left-hand side in a geometric theory of gravity, without introducing any extra fundamental degrees of freedom beyond those that exist in the metric.

Lovelock Gravity

The Lagrangian density from which the field equations are derived is the following:

$$\mathcal{L} = \sqrt{-g} \sum_j \alpha_j \mathcal{R}^j, \tag{1}$$

where $\mathcal{R}^j \equiv \frac{1}{2^j} \delta^{\mu_1 \nu_1 \ldots \mu_{i=1} \eta_i}_{\alpha_1 \beta_1 \ldots \alpha_j \beta_j} \prod^{\alpha_i \beta_i}_{\mu_i \nu_i}$ and $\delta^{\mu_1 \nu_1 \ldots \mu_j \nu_j}_{\alpha_1 \beta_1 \ldots \alpha_j \beta_j} \equiv j! \delta^{\mu_1}_{[\alpha_1} \delta^{\nu_1}_{\beta_1} \ldots \delta^{\mu_j}_{\alpha_j} \delta^{\nu_j}_{\beta_j]}$

- g is the determinant of the metric
- The a_j are a set of arbitrary constants
- $R^{\mu}_{\nu\rho\sigma}$ are the components of the Riemann tensor
- δ^{μ}_{ν} is the Kronecker delta

The tensor $A_{\mu\nu}$ that satisfies properties (i)-(ii) above can be generated from the Lagrangian density in Eq. (1) by integrating it over a region of D-dimensional space- time to construct an action S, and then by varying with respect to the inverse metric $g^{\mu\nu}$

Lovelock Gravity

This gives

$$\delta S = \delta \int_\Omega d^D x \mathcal{L} = \int_\Omega d^D x \sqrt{-g} A_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial\Omega} d^{D-1} x \sqrt{h} B \tag{2}$$

where h is the determinant of the induced metric on $\partial\Omega$ and

$$A^\mu_\nu = -\sum_j \frac{\alpha_j}{2^{j+1}} \delta^{\mu\rho_1\sigma_1\ldots\rho_j\sigma_j}_{\nu\alpha_1\beta_1\ldots\alpha_j\beta_j} \prod_{i=1}^{j} R^{\alpha_i\beta_i}{}_{\rho_i\sigma_i} \tag{3}$$

In dimensions D = 3 and 4 there are two possible terms, corresponding to a constant and to a term of the form $(Riemann)^1$. In fact, this latter term is exactly the Einstein tensor so that in D = 4

$$A^\mu_\nu = -\frac{1}{2}\alpha_0 \delta^\mu_\nu + \alpha_1 \left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R \right) \tag{4}$$

Lovelock Gravity

Continuing to higher dimensions, it can be shown that when $D > 4$ Einstein's equations are not the most general set of field equations that obeys conditions (i)-(ii). In particular, in the case $\mathrm{D} = 5$ or 6 the tensor A^{μ}_{ν} can contain three terms, with the last being order $(Riemann)^2$
This gives a Lagrangian Density of the form

$$\mathcal{L} = \sqrt{-g}\,[\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{G}] \tag{5}$$

where

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{6}$$

is the Gauss Bonnet term.
This tensor provides an alternative set of field equations from those of Einstein, which has no higher than second derivatives of the metric, and which obeys the required symmetry and conservation properties in order for it to be set as being proportional to the stress-energy tensor T^{μ}_{ν}

Einstein - Gauss - Bonnet Gravity

These alternative theories of gravity are of interest partly because string theory predicts that at the classical level Einstein's equations are subject to next-to-leading-order corrections that are typically described by higher-order curvature terms in the action.
The Gauss-Bonnet term is the unique term that is quadratic in the curvature and that results in second-order field equations.
As an example of how Einstein-Gauss-Bonnet gravity arises, it can be shown that M-theory compactified on a Calabi - Yau three-fold down to $D = 5$ takes the effective form

$$S_{eff} = \int d^5x\sqrt{-g}\left(R + \frac{1}{16}c_2^{(I)}V_I\mathcal{G}\right) \tag{7}$$

where $c_2^{(I)}V_I$ depends on the details of the Calabi - Yau manifold.
This is just the 5-D Lovelock theory (5)!

The framework

The vacuum Einstein-Gauss-Bonnet gravity action has the following form,

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{2}\xi(\phi)\mathcal{G}\right), \tag{8}$$

with R denoting the Ricci scalar, $\kappa = \frac{1}{M_p}$ with M_p being the reduced Planck mass. Moreover, $\mathcal{G}$ denotes the Gauss-Bonnet invariant in dimension-4.

We shall assume that the scalar field is solely time-dependent and that the geometry of spacetime is described by a flat Friedman - Robertson - Walker (FRW) metric, with line element,

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}(dx^i)^2\,, \tag{9}$$

where a denotes the scale factor and also for the FRW metric, the Gauss-Bonnet invariant takes the form $\mathcal{G} = 24H^2(\dot{H} + H^2)$.

The framework

The field equations are derived by varying the gravitational action with respect to the metric and to the scalar field, and these are

$$\frac{3H^2}{\kappa^2} = \frac{1}{2}\dot{\phi}^2 + V + 12\dot{\xi}H^3\,, \tag{10}$$

$$\frac{2\dot{H}}{\kappa^2} = -\dot{\phi}^2 + 4\ddot{\xi}H^2 + 8\dot{\xi}H\dot{H} - 4\dot{\xi}H^3\,, \tag{11}$$

$$\ddot{\phi} + 3H\dot{\phi} + V' + 12\xi' H^2(\dot{H} + H^2) = 0\,. \tag{12}$$

By imposing the slow-roll conditions $\dot{H} \ll H^2$, $\frac{\dot{\phi}}{2} \ll V$, $\ddot{\phi} \ll 3H\phi$ and also the constraint $c_T^2 = 1$, where c_T^2 is,

$$c_T^2 = 1 - \frac{Q_f}{2Q_t}\,, \tag{13}$$

and Q_f, F and Q_b defined above are $Q_f = 8(\ddot{\xi} - H\dot{\xi})$, $Q_t = F + \frac{Q_b}{2}$, $F = \frac{1}{\kappa^2}$ and $Q_b = -8\dot{\xi}H$, the field equations are simplified as follows,

The framework

$$H^2 \simeq \frac{\kappa^2 V}{3}\,, \tag{14}$$

$$\dot{H} \simeq -\frac{1}{2}\kappa^2 \dot{\phi}^2\,, \tag{15}$$

$$\dot{\phi} \simeq \frac{H\xi'}{\xi''}\,. \tag{16}$$

The potential of the scalar field and the Gauss-Bonnet scalar coupling function must satisfy the following constraint differential equation,

$$\frac{V'}{V^2} + \frac{4\kappa^4}{3}\xi' \simeq 0\,. \tag{17}$$

The framework

The slow-roll indices for Einstein-Gauss-Bonnet models are,

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \;\; \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \;\; \epsilon_3 = \frac{\dot{F}}{2HF}, \; \epsilon_4 = \frac{\dot{E}}{2HE}, \tag{18}$$
$$\epsilon_5 = \frac{\dot{F} + Q_a}{2HQ_t}, \;\; \epsilon_6 = \frac{\dot{Q}_t}{2HQ_t},$$

with $F = \frac{1}{\kappa^2}$, and also E is defined as follows,

$$E = \frac{F}{\dot{\phi}^2}\left(\dot{\phi}^2 + 3\left(\frac{(\dot{F} + Q_a)^2}{2Q_t}\right) + Q_c\right), \tag{19}$$

where Q_a, Q_t, Q_b and Q_c, and Q_e are

$$Q_a = -4\dot{\xi}H^2, \;\; Q_b = -8\dot{\xi}H, \;\; Q_t = F + \frac{Q_b}{2}, \tag{20}$$
$$Q_c = 0, \;\; Q_e = -16\dot{\xi}\dot{H}.$$

The framework

Employing the simplified equations of motion (14) - (16), the slow-roll indices finally become,

$$\epsilon_1 \simeq \frac{\kappa^2}{2}\left(\frac{\xi'}{\xi''}\right)^2\,, \tag{21}$$

$$\epsilon_2 \simeq 1-\epsilon_1-\frac{\xi'\xi'''}{\xi''^2}\,, \tag{22}$$

$$\epsilon_3 = 0\,, \tag{23}$$

$$\epsilon_4 \simeq \frac{\xi'}{2\xi''}\frac{\mathcal{E}'}{\mathcal{E}}\,, \tag{24}$$

$$\epsilon_5 \simeq -\frac{\epsilon_1}{\lambda}\,, \tag{25}$$

$$\epsilon_6 \simeq \epsilon_5(1-\epsilon_1)\,, \tag{26}$$

The framework

Where,

$$\mathcal{E}(\phi) = \frac{1}{\kappa^2}\left(1 + 72\frac{\epsilon_1^2}{\lambda^2}\right), \;\; \lambda(\phi) = \frac{3}{4\xi''\kappa^2 V} \,. \tag{27}$$

Regarding the observational indices, we have

$$n_{\mathcal{S}} = 1 - 4\epsilon_1 - 2\epsilon_2 - 2\epsilon_4 \,, \tag{28}$$

for the spectral index of the primordial scalar perturbations, while the tensor spectral index is,

$$n_{\mathcal{T}} \simeq -2\epsilon_1 \left(1 - \frac{1}{\lambda} + \frac{\epsilon_1}{\lambda}\right) \,. \tag{29}$$

Finally, the tensor-to-scalar ratio is,

$$r \simeq 16\epsilon_1 \,. \tag{30}$$

The Bottom-up Approach

For our general solution it is essential to express every variable as a function of N, which is the number of e-foldings. To achieve that we write

$$\xi'(\phi) = \frac{dN}{d\phi}\frac{d\xi}{dN}, \tag{31}$$

where the prime,$'$, denotes the differentiation with respect to the scalar field. Using (21), (30) and

$$\frac{\xi''}{\xi'} = \frac{dN}{d\phi}, \tag{32}$$

we derive

$$r(N) = 8\kappa^2 \left(\frac{d\phi}{dN}\right)^2, \tag{33}$$

which is the general form of the scalar to tensor ratio as a function of the number of e-folds, in the Einstein-Gauss-Bonnet theory.

The Bottom-up Approach

The previous expression can be written as,

$$\frac{dN}{d\phi} = \frac{2\kappa\sqrt{2}}{\sqrt{r}}, \tag{34}$$

and thus we obtain a useful expression for ξ' and ξ''

$$\xi' = \frac{dN}{d\phi}\frac{d\xi}{dN} = \frac{2\kappa\sqrt{2}}{\sqrt{r}}\frac{d\xi}{dN}, \tag{35}$$

$$\xi'' = \frac{8\kappa^2}{r}\frac{d^2\xi}{dN^2} - \frac{4\kappa^2}{r^2}\frac{dr}{dN}\frac{d\xi}{dN}. \tag{36}$$

Using the previous equation and the fact that $\xi'' = \frac{dN}{d\phi}\xi'$ we have

$$\xi'' = \frac{8\kappa^2}{r}\frac{d^2\xi}{dN^2} - \frac{4\kappa^2}{r^2}\frac{dr}{dN}\frac{d\xi}{dN} = \frac{2\kappa\sqrt{2}}{\sqrt{r}}\frac{2\kappa\sqrt{2}}{\sqrt{r}}\frac{d\xi}{dN} = \frac{8\kappa^2}{r}\frac{d\xi}{dN}. \tag{37}$$

The Bottom-up Approach

From equation (37) we derive the differential equation of the coupling function with respect to the number of e-folds,

$$\frac{d^2\xi}{dN^2} - \left(\frac{1}{2r}\frac{dr}{dN} + 1\right)\frac{d\xi}{dN} = 0, \tag{38}$$

and its solution is,

$$\xi(N) = C_1 \int \sqrt{r(N)}e^N dN + C_2. \tag{39}$$

From equation (17), the potential of the scalar field takes the following form

$$\frac{1}{V^2}\frac{dV}{d\phi} + \frac{4\kappa^4}{3}\frac{d\xi}{d\phi} = 0. \tag{40}$$

Combining equations (40), (37) we derive to

$$\frac{1}{V^2}\frac{dN}{d\phi}\frac{dV}{dN} + \frac{4\kappa^4}{3}\frac{dN}{d\phi}\frac{d\xi}{dN} = 0. \tag{41}$$

The Bottom-up Approach

Equation (41) is a differential equation of the potential of the scalar field with respect to the number of e-foldings, N. Its solution is

$$V(N) = \frac{3}{4\kappa^4}\frac{1}{\xi(N)}. \tag{42}$$

In this formalism we are able to derive the scalar coupling function, the potential of the scalar field and the spectral indices $n_{\mathcal{S}}$, $n_{\mathcal{T}}$ for various expressions of the tensor-to-scalar ration as a function of the number of e-folds.

Phenomenology of Various Models

In our study the value of the integration constant C_2 did not affect any of the calculated indices, and so we kept $C_2 = 1$ in every model.

Most of our models are of the form of $r = \delta/N^d$ where $d > 0$.

By varying the parameter δ, so that the tensor-to-scalar ratio is within the the Planck 2018 constraints $r < 0.056$, we solved the equation $n_{\mathcal{S}}(C_1) = 0.9649 \pm 0.0042$ for $C_2 = 1$. For each set of the parameters δ, and $n_{\mathcal{S}}$ there are three different C_1, which verify the equation and each of these gives a different $n_{\mathcal{T}}$: a negative one (about -0.04) and two positive (about 0.04 and 0.92) ones.

The last of our models is an exponential model of the form of $r = ae^{-bN}$.

Phenomenology of the $r = \delta/N$ model

The first model that we used was the $r = \delta/N$.
From (39) it follows that the scalar coupling function ξ takes the form,

$$\xi = 2C_1\sqrt{\delta}e^N \,\mathrm{D}(\sqrt{N}) + C_2, \tag{43}$$

Where $\mathrm{D}(x)$ is the Dawson integral, defined as $\mathrm{D}(x) = \exp(-x^2)\int_0^x \exp(t^2)dt$. From (42) it follows that the potential of the scalar field V takes the form,

$$V(N) = \frac{3}{4\kappa^4\left(2C_1\exp(N)\sqrt{\delta}\,\mathrm{D}\left(\sqrt{N}\right) + C_2\right)}, \tag{44}$$

To calculate the scalar spectral index, the corresponding value of the scalar to tensor ratio and the tensor spectral index, we need to calculate the slow roll indices. To do that we use the function $\lambda(N)$ (27), which for this model is,

$$\lambda(N) = \frac{C_2}{8C_1}\sqrt{\frac{\delta}{N}}e^{-N} + \frac{\delta\,\mathrm{D}(\sqrt{N})}{4\sqrt{N}}. \tag{45}$$

Phenomenology of the $r = \delta/N$ model

Using (29) the tensor spectral index takes the form,

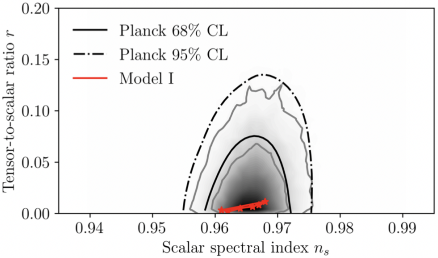
$$n_T = -\frac{2C_2\delta\sqrt{N} + C_1\sqrt{\delta}\exp(N)\left(\delta - 16N4\delta\sqrt{N}\,\mathrm{D}(\sqrt{N})\right)}{16C_2N^{\frac{3}{2}} + 32C_1\sqrt{\delta}\exp(N)N^{\frac{3}{2}}\,\mathrm{D}(\sqrt{N})}. \tag{46}$$

The upper limit of the δ parameter, in order the tensor-to-scalar ratio r to comply with 2018 Planck constrains, is 3.36.

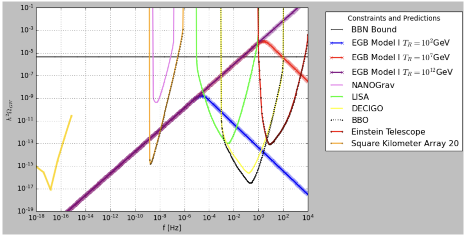
Table: Different values for δ and n_S and the corresponding n_T and r

δ	r	n_S	n_T
0.001	$1.66667 \cdot 10^{-5}$	0.965	0.964288
0.05	0.000833333	0.965	0.964191
0.1	0.00166667	0.965	0.039063
1	0.016667	0.961	0.037994
3	0.05	0.969	0.960455

Using the data presented in the table 1 we constructed the likelihood curve.



(a) Likelihood curves of the $r = \delta/N$ model



(b) The h^2 - scaled primordial gravitational waves energy spectrum for the Model I, versus the sensitivity curves of future primordial gravitational waves experiments.

Phenomenology of the $r = \delta/N^2$ model

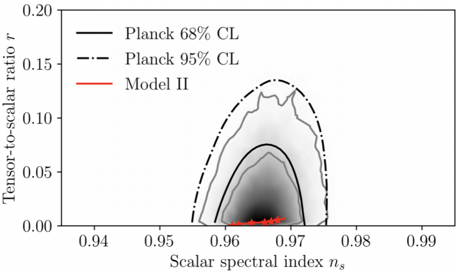
The next model we consider is $r = \frac{\delta}{N^2}$.
The upper limit of the δ parameter, in order the tensor-to-scalar ratio r to comply with 2018 Planck constrains, is 201.6.
The next table shows some of the values of the observational indices we calculated to make the likelihood curve of this model, and the likelihood curve itself

Table: Different values for δ and n_S and the corresponding n_T and r

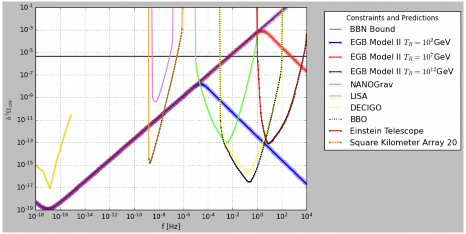
δ	r	n_S	n_T
1	0.000277778	0.961	0.944884
5	0.00138889	0.962	0.945287
10	0.00277778	0.964	0.946191
15	0.00416667	0.967	0.947627
20	0.00555556	0.968	0.947996

Phenomenology of the $r = \delta/N^2$ model

From the previous table we can construct the next diagrams.



(a) Likelihood curves of the $r = \delta/N^2$ model



(b) The h^2 - scaled primordial gravitational waves energy spectrum for the Model II, versus the sensitivity curves of future primordial gravitational waves experiments.

Phenomenology of the $r = \delta/\sqrt{N}$ model

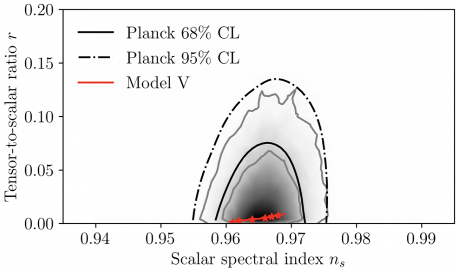
Extrapolating our models to non-integer powers d we consider the model $r = \frac{\delta}{\sqrt{N}}$.
The upper limit of the δ parameter, in order the tensor-to-scalar ratio r to comply with 2018 Planck constrains, is 0.433774.
Some of the different values of the parameter δ, r, the spectral index $n\mathcal{S}$ and n_T for this model are presented in the table:

Table: Different values for δ and $n_{\mathcal{S}}$ and the corresponding n_T and r for Model V

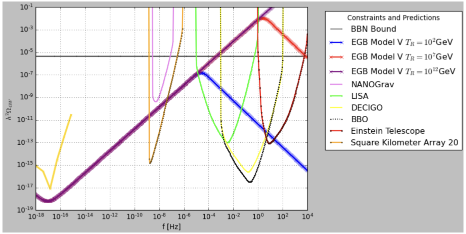
δ	r	$n_{\mathcal{S}}$	n_T
0.01	0.00129099	0.961	0.970617
0.02	0.00258199	0.962	0.970994
0.03	0.00387298	0.964	0.971902
0.04	0.00516398	0.966	0.972808
0.06	0.00774597	0.968	0.97356

Phenomenology of the $r = \delta/\sqrt{N}$ model

Using the above values we constructed the Likelihood curves of this model.



(a) Likelihood curves of the $r = \delta/\sqrt{N}$ model



(b) The h^2 - scaled primordial gravitational waves energy spectrum for the Model V, versus the sensitivity curves of future primordial gravitational waves experiments.

Phenomenology of the Exponential Model

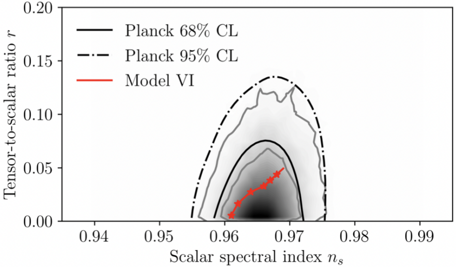
The last model we studied differs from the previous ones as the scalar to tensor ratio is not of the form of $r = \delta/N^d$ with $d > 0$ but of the form of $r = ae^{-bN}$, where a,b are two constant parameters.
In our study the variation of the parameter a affects the value of the scalar to tensor ratio while the variation of the parameter b affects the value of the tensor spectral index.
A set of values of the $(n_{\mathcal{S}}, r, n_{\mathcal{T}})$ with different a,b is shown below.

Table: Different values for a, b and $n_{\mathcal{S}}$ and the corresponding $n_{\mathcal{T}}$ and r

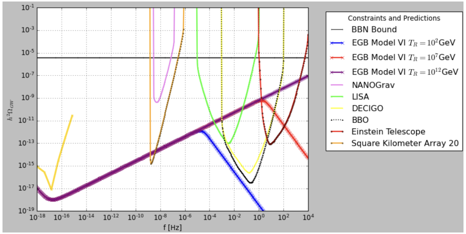
a	b	r	$n_{\mathcal{S}}$	$n_{\mathcal{T}}$
100000000	0.4	0.00377513	0.961	0.538172
100000	0.3	0.001523	0.961	0.658666
100000000	0.43	0.00062402	0.966	0.501779
100000	0.3	0.001523	0.966	0.603673
100000000	0.4	0.00377513	0.969	0.543781

Phenomenology of the Exponential Model

With the values of the above table we construct the diagrams below



(a) Likelihood curves of the $r = ae^{-bN}$ model



(b) The h^2 - scaled primordial gravitational waves energy spectrum for the Exponential Model , versus the sensitivity curves of future primordial gravitational waves experiments.

Conclusions

- We thoroughly studied several models of interest and we showed that the compatibility with the Planck 2018 data can be achieved for a general range of the free parameters.

- Most of the models lead to a blue-tilted tensor spectral index and as we showed, the energy spectrum of the inflationary primordial gravitational waves can be detectable by most of the future experiments on gravitational waves.

- For most of the models, the tensor-to-scalar ratio can take significantly smaller values than most of the R^2-like scalar field models.

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