New Inflationary solution from old inflation

Andronikos Paliathanasis

Durban University of Technology
Universidad Austral de Chile

Cosmic Inflation @ A.U.Th., Thessaloniki, 2022

FLRW geometry

Spatially flat FLRW spacetime

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{1}$$

Field equaitons in General Relativity with matter source

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}\ ;\ \ T^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

where $R = 6\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right)$,

$$R^0_0 = 3\left(\frac{\dot{a}}{a}\right)^2,\ R^A_A = 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} \tag{3}$$

and

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right). \tag{4}$$

Scalar field cosmology

For comoving observers $(u^{\mu} = \delta^{\mu}_{0})$ and for a FLRW spacetime, the Einstein field equations are

$$H^2 = \frac{\kappa}{3}\left(\rho_m + \rho_\phi\right)$$

$$3H^2 + 2\dot{H} = -\kappa(P_m + P_\phi)$$

where $H(t) \equiv \dot{a}/a$ is the Hubble function.
Furthermore, assuming that the scalar field and matter do not interact, we have the two following equations

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

while the corresponding equation of state (EoS) parameters are given by $w_m = P_m/\rho_m$ and $w_\phi = P_\phi/\rho_\phi$, where

$$\rho_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi)\ ,\ \ P_\phi \equiv \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

Point-like Lagrangian

The Lagrangian which provides all the field equations from the variational principle is a singular Lagrangian of the form

$$L = \frac{1}{2N(t)}\gamma_{\mu\nu}(q)\dot{q}^{\mu}(t)\dot{q}^{\nu}(t) - N(t)\,U(q)$$

where the Hamiltonian is also the contraint equation $\frac{\partial L}{\partial N} = 0$, that is,

$$N\mathcal{H} = N\left(\frac{1}{2}\gamma^{\mu\nu}p_{\mu}p_{\nu} + 1\right) \approx 0$$

Integrability

It is well known that there exists a unique relation between the symmetries of this kind of systems of differential equations with the symmetries which define the underlying geometry . That means that any generator of a symmetry vector for the dynamical system has to be a symmetry also for the geometry. For instance the conservation law of momentum for the free particle follows from the translation symmetry of the Euclidean spacetime. The group of translations with the group of rotations form the group of isometries or Killing vectors of the Euclidean space. By definition a Killing vector in a Riemannian manifold is the generator of the transformation which keep invariant the length and the angles. On the other hand, a Homothetic vector is the generator of the transformation which keep invariant the angles and rescale by a constant the length, whereas a Conformal vector is called the generator of the transformation which preserves the angles on the space.

Integrability

- Now for autonomous Hamiltonian systems the "Energy" denotes the volume in the phase space. For any isometry which leave invariant this volume in the phase space corresponds a conservation law which commutes with the Hamiltonian. As far as concerns the Homothetic vector, the solutions can be transformed under other solution but with a rescaled "Energy" value. These two transformations relates objects which are congruent, with the identical congruent to be provided by the isometries.
- The situation is totally different under conformal transformations. Indeed Hamiltonian systems are not invariant under conformal transformations except if the "Energy" is zero, which means that the volume in the phase space has dimensions zero. Moreover the volume continues to be zero under conformal transformations and consequently conservation laws can be constructed.

Integrability

- In order to demonstrate that mathematically, consider $\mathcal{H}(\mathbf{p},\mathbf{q}) = 0$ to be the energy of an autonomous Hamiltonian system and $I(\mathbf{p},\mathbf{q})$ be a conservation law generated by a conformal vector. Then it follows that there exists a function, ω, such that $D_t(I) = I_{,t} + \{I,\mathcal{H}\} = \omega\mathcal{H}$; that is, $D_t(I) = 0$, which means that I is a conservation law. These kinds of conservation laws are generated by nonlocal symmetries, which reduce to local when $\omega = const$ or $\omega = 0$.

Integrability

- Because of the constraint equation we can say that the Energy of the Mechanical analogue is zero and construct conservation laws by using the conformal algebra of the minisuperspace. In particular, for every Conformal vector field there corresponds a conservation law for the field equations, for any function, $V(\phi)$. Moreover, because the minisuperspace has dimension two, it admits an infinite-dimensional conformal algebra, that is, there exists an infinitenumber of (nonlocal) conservation laws. Of course these conservation laws are not in involution with each other, but they are with the Hamiltonian applying the constraint equation, $\mathcal{H}(\mathbf{p}, \mathbf{q}) = 0$.

Summarize

- For Lagrangians of that forms the nonlocal conservation laws are generated by the Conformal Killing vectors of the minisuperspace.
- The minisuperspace in scalar field cosmology has dimension two, which means that admits infinity number of CKVs and nonlocal conservation laws.
- The existence of a nonlocal conservation law plus the Constraint equation is sufficient to prove the integrability.

General Solution

In the case of a spatially flat universe, $K = 0$, and without matter source, $\rho_{m0} = 0$, it has been found that [Dimakis et al. PRD (2016)]

$$\phi(\omega) = \pm\frac{\sqrt{6}}{6}\int\sqrt{F'(\omega)}d\omega,\ \ V(\omega) = \frac{1}{12}e^{-F(\omega)}\left(1 - F'(\omega)\right) \tag{5}$$

and

$$\rho_{\phi}(\omega) = \frac{1}{12}e^{-F(\omega)}\ ,\ \ P_{\phi}(\omega) = \frac{1}{12}e^{-F(\omega)}\left(2F'(\omega) - 1\right). \tag{6}$$

where the spacetime is

$$ds^2 = -e^{F(\omega)}d\omega^2 + e^{\omega/3}(dx^2 + dy^2 + dz^2). \tag{7}$$

That is analytical solution for arbitrary potential. The form of the potential fixes the EoS and provides an first order-differential equation $p_{\phi}\left(\omega\right) = \Phi\left(\rho_{\phi}\right)$ which can be reduced to an algebraic equation.

General Solution

With the presence of a perfect fluid (or curvature) the analytical solution is

$$\phi(\omega) = \pm\frac{\sqrt{6}}{6}\int\left[\left(F'(\omega) - 6\gamma\rho_{m0}e^{F-\frac{\gamma}{2}\omega}\right)\right]^{1/2}d\omega, \tag{8}$$

where now

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}\left(1 - F'(\omega)\right) + \frac{\gamma}{2}\rho_{m0}\,e^{-\frac{\gamma}{2}\omega} \tag{9}$$

and the fluid components become $\rho_\phi = \frac{1}{12}e^{-F(\omega)} - \rho_{m0}\,e^{-\frac{\gamma}{2}\omega}$ and $P_\phi = \frac{1}{12}e^{-F(\omega)}\left(2F'(\omega) - 1\right) - (\gamma - 1)\,\rho_{m0}\,e^{-\frac{\gamma}{2}\omega}$.

Slow-Roll parameters

The potential slow-roll parameters (PSR)

$$\varepsilon_V = \left(\frac{V_{,\phi}}{2V}\right)^2 \ , \ \eta_V = \frac{V_{,\phi\phi}}{2V},$$

provide with an inflationary universe when $\varepsilon_V << 1$. The condition $\eta_V << 1$ is also important for the duration of the inflation phase.
Alternatively, more accurate parameters are the Hubble slow-roll parameters (HSR)

$$\varepsilon_H = -\frac{d\ln H}{d\ln a} = \left(\frac{H_{,\phi}}{H}\right)^2 , \ \eta_H = -\frac{d\ln H_{,\phi}}{d\ln a} = \frac{H_{,\phi\phi}}{H}.$$

in which

$$\varepsilon_V = \varepsilon_H\left(\frac{3-\eta_H}{3-\varepsilon_H}\right)^2 \ , \ \eta_H = \frac{\sqrt{\varepsilon_H}}{3-\varepsilon_H}\eta_{H,\phi} + \left(\frac{3-\eta_H}{3-\varepsilon_H}\right)(\varepsilon_H+\eta_H)$$

Slow-Roll parameters

The HSR parameters can be expressed in terms of ω, and function $F(\omega)$ as

$$\varepsilon_H = 3F' \ , \ \eta_H = 3\frac{(F')^2 - F''}{F'},$$

while for the Number of e-folds

$$N_e = \int_{t_i}^{t_f} H(t)\, dt = \ln \frac{a_f}{a_i} = \frac{1}{6}(\omega_f - \omega_i)\,,$$

Finally for higher-order corrections we calculate

$$\xi_H \equiv \frac{H_\phi H_{\phi\phi\phi}}{H^2} = -\frac{9\sqrt{6}}{4(F')^{\frac{5}{2}}}\left[(F')^4 - \left(3F'^2 + 2F''\right)F'' + 2F'F'''\right].$$

Spectral Indices

From the recent data analysis by the Planck 2018 collaboration, it was found that the value of the spectral index for the density perturbations is $n_s = 0.9649 \pm 0.0042$, while the range of the scalar spectral index is $n_s' = -0.005 \pm 0.013$. The tensor to scalar ratio, r, has been found to have a value smaller than 0.10, i.e., $r < 0.10$.

The mathematical expression which relates the HSR parameters to the spectral indices n_s in the first approximation is

$$n_s \equiv 1 - 4\varepsilon_H + 2\eta_H,$$

while the tensor to scalar ratio is $r = 10\varepsilon_H$. Moreover, in the second approximation the spectral index, n_s, becomes

$$n_s \equiv 1 - 4\varepsilon_H + 2\varepsilon_H - 8\left(\varepsilon_H\right)^2\left(1 + 2C\right) + \varepsilon_H\eta_H\left(10C + 6\right) - 2C\xi_H,$$

where $C = \gamma_E + \ln 2 - 2 = -0.7296$. Finally, the running index is given to be

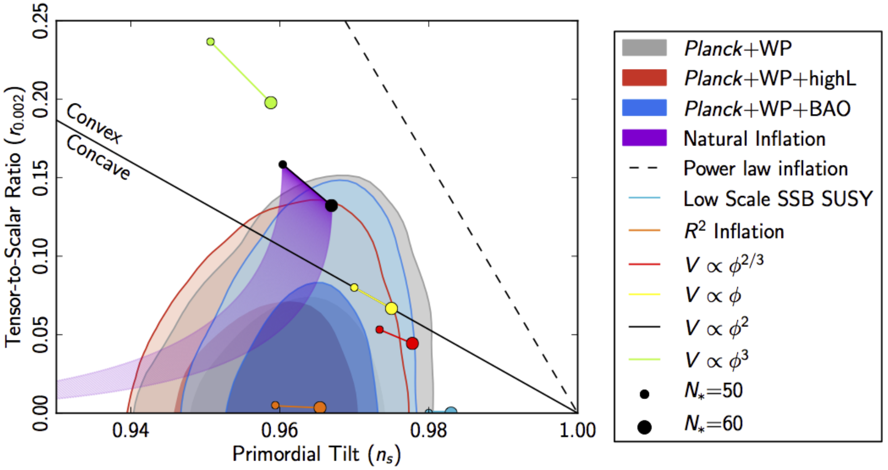
$$n_s' \equiv 2\varepsilon_H\eta_H - 2\xi_H.$$

Reconstruct the inflationary potential

The construction of the inflaton scalar field potential from observational data is an open problem of special interest.

- Perturbative reconstruction approach [Lidsey et al. Rev. M.P (1997) and references therein]
- Stochastic perturbative approach [Easther et al. PRD (2003)]
- Other methods [Starobinsky JETP Lett. (2005);Wohns et al JCAP (2011);Urena-Lopez PRD (2016)]
- Closed-form inflationary potentials by using the slow-roll parameters $\varepsilon_\phi, \eta_\phi$ have been derived before [Vallinoto et al. PRD (2004);Chiba PTEP (2015);Lin et al. MNRAS (2016)]

Planck Collaboration



Reconstruct the inflationary potential

We express the spectral indices in terms of the Hubble slow-roll parameters and we assume that in the first-order approximation

$$n_s - 1 = h(r)$$

For the function $h(r)$, we consider

- $h(r)$ is constant, $h(r) = -2n_0$,
- $h(r)$ is linear, $h(r) = n_1 r - 2n_0$
- $h(r)$ is quadratic, $h(r) = n_2 r^2 + n_1 r - 2n_0$

From this, we derive second-order equations whose solutions provide us with the explicit forms for the expansion scale-factor, the scalar-field potential, and the effective equation of state for the scalar field

Constant index ns

The master equation is

$$F'' + \left(F'\right)^2 - \frac{n_0}{3}F' = 0, \tag{10}$$

which gives $F(\omega) = \ln\left(F_1 \exp\left(\frac{n_0}{3}\omega\right) + F_0\right)$ and perfect fluid

$$p_\phi = A\rho_\phi^2 + B\rho_\phi \tag{11}$$

or $F(\omega) = \ln\left(F_1\left(\omega - \omega_0\right)\right)$, $n_0 = 0$, and perfect fluid

$$p_\phi = \gamma\rho_\phi^2 - \rho_\phi \text{ with } \lambda = 2, \tag{12}$$

Recall that $n_0 = 0$ is the Harrison-Zeldovich spectrum.

Linear function

The master equation is

$$F'' + (1 - n_1)\left(F'\right)^2 - \frac{n_0}{3}F' = 0, \tag{13}$$

which gives

$$p_\phi = A\rho_\phi^\lambda + B\rho_\phi\ ,\ \lambda = 2 - n_1$$

and scalar field potential

$$V(\phi) \propto \left(2A + (B-1)\sinh^2\left(\frac{\sqrt{3(1+B)}(\lambda-1)}{2}\phi\right)\right) \times \times \left(\sinh^2\left(\frac{\sqrt{3(1+B)}(\lambda-1)}{2}\phi\right) - A\right)^{\frac{\lambda-2}{\lambda-1}}$$

Quadratic function

The master equation is

$$F'' + 3n_2\left(F'\right)^3 + (1-n_1)\left(F'\right)^2 - \frac{n_0}{3}F' = 0, \tag{14}$$

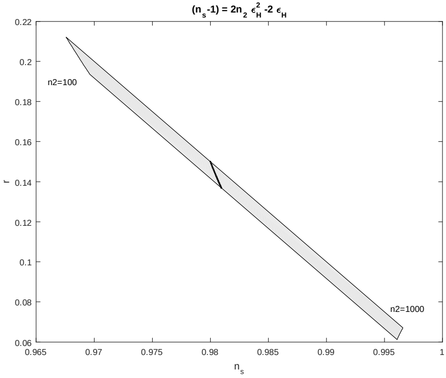
In the limit in which $n_1 = 1$,

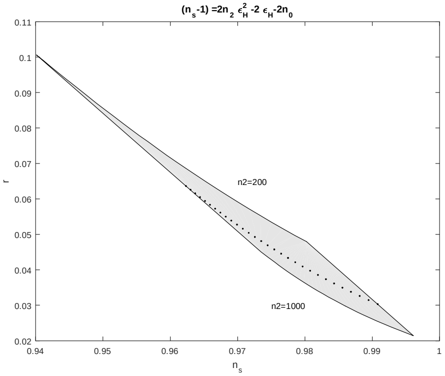
$$p_\phi = \left(\frac{6\rho_\phi}{n_2 \ln\left(12\rho_\phi\right)} - \rho_\phi\right) \ , \ n_0 = 0. \tag{15}$$

On the other hand, for $n_0 \neq 0$ we find that that the EoS is

$$p_\phi = 2\sqrt{\frac{n_0}{n_2}}\left(\frac{16n_0 n_2 \rho_\phi^2 + 1}{16 n_0 n_2 \rho_\phi^2 - 1}\right) - \rho_\phi \ , \ n_0 \neq 0. \tag{16}$$

Constraints





Escape from Inflation

We rewrite the master equation terms of $\varepsilon_H(\omega)$ as follows

$$3\varepsilon'_H = \left(n_0 + (n_1 - 1)\,\varepsilon_H - n_2\varepsilon_H^2\right)\varepsilon_H. \tag{17}$$

This equation has the following critical points $\varepsilon_H^{(0)} = 0$, $\varepsilon_H^{(\pm)} = \frac{n_1 - 1 \pm \sqrt{(1-n_1)^2 + 4n_0 n_2}}{2n_2}$,for $n_2 \neq 0$,or $\varepsilon_H^{(0)} = 0$, $\varepsilon_H^{(1)} = \frac{n_0}{1-n_1}$,when $n_2 = 0$ and $n_1 \neq 1$.Hence, in order for inflation to end in the cosmological models that we studied, the free parameters of the models have to be constraineded so that one of the critical points, $\varepsilon_H^{(\pm)}$ or $\varepsilon_H^{(1)}$, is attractor, and also that $\varepsilon_H^{(\pm)} \geq 1$ or $\varepsilon_H^{(1)} \geq 1$. We note that point $\varepsilon_H^{(0)}$ describes a de Sitter universe (that is, $w_\phi = -1$), while the rest of the critical points the equation of state parameter, w_ϕ, is constant. Therefore, from the previous analysis we see that at the critical points the scalar field potential is described by the exponential function.

Escape from Inflation

For $n_2 \neq 0$, a necessary condition for an exit from the inflation to occur, is that the critical points $\varepsilon_H^{(\pm)}$ are real; that is, $4n_0 n_2 \geq -\frac{(1-n_1)^2}{4}$. In the special limit in which $n_0 = 0$, the points $\varepsilon_H^{(\pm)}$ reduce to $\varepsilon_H^{(0)}$ and $\varepsilon_H^{(2)} = \frac{n_1 - 1}{n_2}$. In that case, the two points are stable when $n_2 > 0$, and $\varepsilon_H^{(2)}$ is positive for any value of $n_1 > 1$. However, in the general scenario with $n_0 \neq 0$, it follows easily that in order for $\varepsilon_H^{(0)}$ to be an elliptic point we require $n_0 > 0$. Moreover, by assuming the condition $\varepsilon_H^{(\pm)} > 1$, we find that only the point $\varepsilon_H^{(+)}$ can be an attractor outside the inflationary era and this is possible only when the free parameters satisfy the conditions (i) $n_2 < 0$, $n_1 < 1 + 2n_2$, $n_0 > 1 - n_1 + n_2$ and $4n_0 n_2 \geq -\frac{(1-n_1)^2}{4}$, (ii) $n_2 > 0$, $n_1 > 1 + n_2$ and $n_0 > 1 - n_1 + n_2$, or (iii) $n_2 > 0$, $n_1 \leq 1 + n_2$ and $n_0 > 0$. Therefore, for values of the free parameters in those ranges only the third model, i.e. with $h(r)$ is a quadratic function, admits an attractor outside the inflationary era.

The sl(3,R) algebra

The master equations

$$F'' + (F')^2 - \frac{n_0}{3}F' = 0, \tag{18}$$

$$F'' + (1 - n_1)(F')^2 - \frac{n_0}{3}F' = 0, \tag{19}$$

$$F'' + 3n_2 (F')^3 + (1 - n_1)(F')^2 - \frac{n_0}{3}F' = 0, \tag{20}$$

are algebraic equivalent.

The sl(3,R) algebra

Consider the first with $n_0 = 0$, and

$$F(\omega) \rightarrow (1 - n_1) \bar{F}(\omega), \tag{21}$$

then the equation becomes

$$\bar{F}'' + (1 - n_1) \left(\bar{F}'\right) = 0$$

while the spacetime is transformed

$$ds^2 = -\left(e^{-\bar{F}(\omega)}\right)^{(1-n_1)} d\omega^2 + e^{\omega/3}(dx^2 + dy^2 + dz^2). \tag{22}$$

The existence of these kinds of transformations which transform the one model into another is not a coincidence. The master equations are maximally symmetric. In particular they are invariant under the action of one-parameter point transformations (Lie point symmetries) which form the $sl(3, R)$ Lie algebra.

The sl(3,R) algebra

Famous maximally symmetric equations: The free particle $y'' = 0$ and the "oscillator" $y'' + \omega(t)\, y = 0$.

Darboux transformation is on that class of transformations.

Consider now the classical Newtonian analogue of a free particle and an observer whose measuring instruments for time and distance are not linear. By using the measured data of the observer we reach in the conclusion that it is not a free particle. On the other hand, in the classical system of the harmonic oscillator an observer with nonlinear measuring instruments can conclude that the system observed is that of a free particle, or that of the damped oscillator or another system. From the different observations, various models can be constructed. However, all these different models describe the same classical system and the master equations are invariant under the same group of point transformations but in different parametrization.

In the master equations that we studied there is neither position nor time variables: the independent variable is the scale factor $\omega = 6 \ln a$, and the Hubble function is the dependent variable, $H(a)$. Therefore, we can say that at the level of the first-order approximation for the spectral indices, various representations of the variables $\{a, H(a)\}$ provide different observable values for the spectral indices. This property is violated when we consider the 2nd-order approximation.

Thank you for your attention.