Cosmic Inflation: From Observations to Particle Models

Cosmological Hyperfluids, Torsion and Non-metricity

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Outline

- Non-Riemannian Geometry: Conventions/Notation
- Hyperfluids, Torsion and Non-metricity in Cosmology
- Perfect Hyperfluid: The Foundation
- Application: Friedmann Eqs with Torsion and Non-Metricity
- Conclusions/Further Prospects

The talk is based on my paper
"Cosmological Hyperfluids, Torsion and Non-metricity" Published in: Eur.Phys.J.C 80 (2020) 11, 1042 • e-Print: 2003.07384 [gr-qc]
and also on
"The Perfect Hyperfluid of Metric-Affine Gravity: The Foundation" Published in: JCAP 04 (2021) 072 • e-Print: 2101.07289 [gr-qc]

Metric-Affine Gravity

Metric Gravity

- $\Gamma^{\alpha}{}_{\mu\nu} \rightarrow$ *torsionless* , metric compatibility $\nabla_{\sigma} g_{\mu\nu} = 0$
- $S = S_{Gravity} + S_{Matter} = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Phi)\right]$

Teleparallel/Symmetric Teleparallel Gravity

- $R^{\alpha}{}_{\beta\mu\nu} = 0$, $\nabla_{\sigma} g_{\mu\nu} = 0$ but $S_{\mu\nu}{}^{\alpha} = \Gamma^{\alpha}{}_{[\mu\nu]} \neq 0$
- $R^{\alpha}{}_{\beta\mu\nu} = 0$, $S_{\mu\nu}{}^{\alpha} = 0$ but $Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} \neq 0$

Metric-Affine Gravity (MAG)

- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}, \Phi)\right] \Rightarrow$ No a priori constraints on the geometry.

Geometrical Objects

Two distinctively different notions on a manifold

- Metric Tensor $g_{\mu\nu}$: Defines distances,lengths and dot products

$$||\alpha||^2 := \alpha^\mu \alpha^\nu g_{\mu\nu} \ , \ \ (\alpha \cdot \beta) := \alpha^\mu \beta^\nu g_{\mu\nu}$$

- Affine-Connection $\Gamma^\lambda{}_{\mu\nu}$: Defines parallel transport of tensor fields on the manifold

$$\nabla_\lambda u^\mu = \partial_\lambda u^\mu + \Gamma^\mu{}_{\nu\lambda} u^\nu$$

The two need not be related a priori! Their relation may be found after solving the field equations!

Geometrical Objects

Torsion

- $\nabla_{[\mu}\nabla_{\nu]}\phi = S_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi$, Torsion Tensor $S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}$

Curvature

- $[\nabla_{\alpha}, \nabla_{\beta}]u^{\mu} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu} + 2S_{\alpha\beta}{}^{\nu}\nabla_{\nu}u^{\mu}$
  Curvature Tensor: $R^{\mu}{}_{\nu\alpha\beta} := 2\partial_{[\alpha}\Gamma^{\mu}{}_{|\nu|\beta]} + 2\Gamma^{\mu}{}_{\rho[\alpha}\Gamma^{\rho}{}_{|\nu|\beta]}$

Non-Metricity

- $Q_{\alpha\mu\nu} := -\nabla_{\alpha}g_{\mu\nu} = -\partial_{\alpha}g_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\nu} + \Gamma^{\lambda}{}_{\nu\alpha}g_{\lambda\mu}$

Contractions

Contractions of Riemann

- Ricci Tensor: $R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}$
- Homothetic Curvature: $\widehat{R}_{\alpha\beta} := R^{\mu}{}_{\mu\alpha\beta}$
- 2nd Ricci Tensor: $\check{R}_{\mu\nu} := R_{\mu\alpha\beta\nu}g^{\alpha\beta}$
- Ricci Scalar: $R := R_{\mu\nu}g^{\mu\nu} = -\check{R}_{\mu\nu}g^{\mu\nu}$

Torsion/Non-metricity related vectors

$$S_{\mu} = S_{\mu\lambda}{}^{\lambda}\ , \qquad \tilde{S}^{\mu} = \epsilon^{\mu\nu\rho\sigma}S_{\nu\rho\sigma}\ \ (only\ for\ n=4)$$

$$Q_{\mu} = g^{\alpha\beta}Q_{\mu\alpha\beta}\ , \qquad \tilde{Q}_{\mu} = g^{\rho\alpha}Q_{\rho\alpha\mu}$$

Affine Connection

Affine connection decomposition

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{1}{2} g^{\alpha\lambda}(Q_{\mu\nu\alpha} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda}(S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha})$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$ is the Levi-Civita part of the connection. Distortion: $N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu}$

Decompositions

Each quantity $\Rightarrow$ decomposed into Riemannian and non-Riemannian counterparts. Example:

$$R = \tilde{R} + S_{\mu\nu\alpha} S^{\mu\nu\alpha} - 2 S_{\mu\nu\alpha} S^{\alpha\mu\nu} - 4 S_\mu S^\mu - 4 \tilde{\nabla}_\mu S^\mu$$
$$+ \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} - \frac{1}{4} Q_\mu Q^\mu + \frac{1}{2} Q_\mu \tilde{Q}^\mu$$
$$+ 2 Q_{\alpha\mu\nu} S^{\alpha\mu\nu} + 2 S_\mu (\tilde{Q}^\mu - Q^\mu) + \tilde{\nabla}_\mu (\tilde{Q}^\mu - Q^\mu - 4 S^\mu)$$

Hypermomentum, Canonical and Metrical Energy Momentum Tensors

Metrical and Canonical Energy Momentum Tensor

Metrical: $T_{\alpha\beta} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}$. Canonical: $t^{\mu}{}_{c} = \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta e_{\mu}{}^{c}}$

Hypermomentum Tensor

Hypermomentum: $\Delta_{\lambda}{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$

Relation Between Energy Tensors

$$t^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} - \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu})$$

where $\hat{\nabla}_{\nu} = 2S_{\nu} - \nabla_{\nu}$.

Conservation Laws

Working in exterior calculus from the GL and diff invariance we get

From GL

$$t^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} - \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu})$$

From Diff

$$\frac{1}{\sqrt{-g}}\hat{\nabla}_{\mu}(\sqrt{-g}t^{\mu}{}_{\alpha}) = -\frac{1}{2}\Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha} + \frac{1}{2}Q_{\alpha\mu\nu}T^{\mu\nu} + 2S_{\alpha\mu\nu}t^{\mu\nu}$$

From Diff using coordinates

$$\sqrt{-g}(2\tilde{\nabla}_{\mu}T^{\mu}{}_{\alpha} - \Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha}) + \hat{\nabla}_{\mu}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\alpha}{}^{\mu\nu})$$
$$+2S_{\mu\alpha}{}^{\lambda}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}) = 0$$

Homogeneous Cosmology with Torsion and non-metricity

- Applying Cosmological Principle to Torsion [Tsamparlis,1979]:
  $S_{01}{}^{1} = S_{02}{}^{2} = S_{03}{}^{3} = \dots = S_{0m}{}^{m} \neq 0 \quad (\textit{no sum})$
  $S_{ijk} \propto \epsilon_{ijk} \neq 0$ (only for $n = 4$)
- Applying it to Non-Metricity[Minkevich,1998]:
  $Q_{011} = \dots = Q_{0mm} \neq 0,\ Q_{110} = \dots = Q_{mm0} \neq 0,$
  $Q_{000} \neq 0$ Here $m = n - 1$=spatial space dim
  $\Longrightarrow$ The rest vanish!

Covariant Forms

The covariant forms of the above read [D.I,2020]

- $S^{(n)}_{\mu\nu\alpha} = 2u_{[\mu}h_{\nu]\alpha}\Phi(t) + \epsilon_{\mu\nu\alpha\rho}u^{\rho}P(t)\delta_{n,4}$
- $Q_{\alpha\mu\nu} = A(t)u_{\alpha}h_{\mu\nu} + B(t)h_{\alpha(\mu}u_{\nu)} + C(t)u_{\alpha}u_{\mu}u_{\nu}, \quad \forall$ n

$N^{(n)}_{\alpha\mu\nu} = X(t)u_{\alpha}h_{\mu\nu} + Y(t)u_{\mu}h_{\alpha\nu} + Z(t)u_{\nu}h_{\alpha\mu} + V(t)u_{\alpha}u_{\mu}u_{\nu} + \epsilon_{\alpha\mu\nu\lambda}u^{\lambda}W(t)\delta_{n,4}$ for the distortion.

Isotropic Hypermomentum

Imposing Cosm. Principle to Hypermomentum($\mathcal{L}_{\xi^i}\Delta_{\alpha\mu\nu}=0$)
$\Delta_{i00}=\Delta_{0i0}=\Delta_{00i}=0$,
$\Delta_{110}=...=\Delta_{mm0}$,$\Delta_{011}=...=\Delta_{0mm}$(no sum)

Covariant Form of Hypermomentum

Using an $1+(n-1)$ split we get the covariant form[D.I,2020]:

- $\Delta^{(n)}_{\alpha\mu\nu}=\phi h_{\mu\alpha}u_\nu+\chi h_{\nu\alpha}u_\mu+\psi u_\alpha h_{\mu\nu}+\omega u_\alpha u_\mu u_\nu+\delta_{n,4}\epsilon_{\alpha\mu\nu\kappa}u^\kappa\zeta$

Most General form of Hypermomentum respecting isotropy!

Comments

1. In an FLRW $\phi,\chi,...$ depend only on time t. If homogeneity is relaxed $\phi=\phi(t,x^i)$ etc. (more about it later)
2. Hypermomentum generally contributes 5 dof in a Cosmological setting($n=4$). (and 4 dof for $n\neq 4$).

Hypermomentum Decomposition (Matter with Microstructure)

- Spin Part: $\Delta_{[\alpha\mu]\nu} = (\psi - \chi) u_{[\alpha} h_{\mu]\nu} + \delta_{n,4} \epsilon_{\alpha\mu\nu\kappa} u^{\kappa} \zeta$
- Dilation Part: $\Delta_{\nu} := \Delta_{\alpha\mu\nu} g^{\alpha\mu} = \Big[(n-1)\phi - \omega \Big] u_{\nu}$
- Shear Part: $\breve{\Delta}_{\alpha\mu\nu} = \Delta_{(\alpha\mu)\nu} - \frac{1}{n} g_{\alpha\mu} \Delta_{\nu} = \frac{(\phi+\omega)}{n} \Big[h_{\alpha\mu} + (n-1) u_{\alpha} u_{\mu} \Big] u_{\nu} + (\psi + \chi) u_{(\mu} h_{\alpha)\nu}$

Sourcing Torsion and Non-Metricity ($5 = 2 + 3$)

By means of the connection field eqs, the above parts act as sources producing spacetime torsion and non-metricity (see example later).

The Perfect (Ideal) Hyperfluid [D.I, 2020]

Energy Momentum:

$T_{\mu\nu}=t_{\mu\nu}=\rho u_\mu u_\nu + p h_{\mu\nu}$

Hypermomentum :

$\Delta^{(n)}_{\alpha\mu\nu}=\phi h_{\mu\alpha}u_\nu+\chi h_{\nu\alpha}u_\mu+\psi u_\alpha h_{\mu\nu}+\omega u_\alpha u_\mu u_\nu+\delta_{n,4}\epsilon_{\alpha\mu\nu\kappa}u^\kappa\zeta$

Conservation laws (obtained from diff invariance)

$\tilde{\nabla}_\mu T^\mu{}_\nu=\frac{1}{2}\Delta^{\alpha\beta\gamma}R_{\alpha\beta\gamma\nu}.\quad \hat{\nabla}_\nu\left(\sqrt{-g}\Delta_\lambda{}^{\mu\nu}\right)=0$

We call it hypermomentum preserving.

Note

The conservation law for hypermomentum (2nd eq. above) in an FLRW Universe really contains 2 independent eqs for the 5 fields.
$\Rightarrow$ 3 eqs of state must be provided.

Simple Example:

Consider the Metric-Affine Theory
$S = \frac{1}{2\kappa}\int d^n x\sqrt{-g}R(g,\Gamma) + S_M[g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \Psi]$ where S_M is Perfect Hyperfluid Matter.

- g-variation: $R_{(\mu\nu)} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$
- Γ-variation: $-\frac{\nabla_\lambda(\sqrt{-g}g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\sigma(\sqrt{-g}g^{\mu\sigma})\delta^\nu_\lambda}{\sqrt{-g}}$
  $+2(S_\lambda g^{\mu\nu} - S^\mu\delta^\nu_\lambda + g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu}) = \kappa\Delta_\lambda{}^{\mu\nu}$

Homogeneous Cosmology

Consider a flat FLRW background with the usual line element
$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \Rightarrow$ Get modified Friedmann eqs in the presence torsion and non-metricity (induced by Hypermomentum).

Connecting them to their sources

Using the connection field eqs we can express the torsion and non-metricity functions in terms of their sources (hypermomentum components)

$$A=\kappa(\phi-\chi-\psi)\ ,\quad B=-2\kappa\phi\ ,\quad C=-\kappa\omega$$

$$\Phi=\frac{\kappa}{4}\left[\frac{1}{(n-2)}\Big((n-1)\psi+(n-3)\chi\Big)-2\phi\right]\ ,\quad P=-\frac{\kappa}{2}\zeta$$

Note

This is most easily achieved by first writing

$$N^{(n)}_{\alpha\mu\nu}=X(t)u_\alpha h_{\mu\nu}+Y(t)u_\mu h_{\alpha\nu}+Z(t)u_\nu h_{\alpha\mu}$$
$$+V(t)u_\alpha u_\mu u_\nu+\epsilon_{\alpha\mu\nu\lambda}u^\lambda W(t)\delta_{n,4}$$

relate $X, Y, \dots$ to $\phi, \chi, \dots$ then use $Q_{\nu\alpha\mu}=2N_{(\alpha\mu)\nu}$, $S_{\mu\nu\alpha}=N_{\alpha[\mu\nu]}$

Friedmann Eqs with Torsion and Non-metricity

$$
\begin{aligned}
H^2 = {} & \frac{2\kappa}{(n-1)(n-2)}\rho + \frac{2}{(n-2)}P^2\delta_{n,4} \\
& -\frac{H}{(n-2)}\left[(n-1)X-(n-3)Y+A+C\right] \\
& -\frac{1}{(n-2)}(\dot{X}+\dot{Y}) - \frac{1}{2(n-2)}(X-Y)(A+C)+XY
\end{aligned}
$$

$$
\begin{aligned}
\frac{\ddot{a}}{a} = {} & -\frac{\kappa}{(n-1)(n-2)}\Big[(n-3)\rho+(n-1)p\Big] \\
& +\dot{Y}+H\Big(Y+\frac{A}{2}+\frac{C}{2}\Big)-\frac{Y}{2}(A+C)
\end{aligned}
$$

where $Y=2\Phi+\frac{A}{2}$, $X=\frac{B}{2}-2\Phi-\frac{A}{2}$.

Conservation Laws

The previous Friedmann eqs are subject to the conservation Laws of the Perfect Cosmological Hyperfluid (PCH):

- $\dot{\rho}+(n-1)H(\rho+p)=-\frac{1}{2}u^{\mu}u^{\nu}(\chi R_{\mu\nu}+\psi\check{R}_{\mu\nu})$
- $-\delta^{\mu}_{\lambda}\frac{\partial_{\nu}(\sqrt{-g}\phi u^{\nu})}{\sqrt{-g}}-u^{\mu}u_{\lambda}\frac{\partial_{\nu}\Big(\sqrt{-g}(\phi+\chi+\psi+\omega)u^{\nu}\Big)}{\sqrt{-g}}$
  $+\left[\left(2S_{\lambda}+\frac{Q_{\lambda}}{2}\right)u^{\mu}-\nabla_{\lambda}u^{\mu}\right]\chi+$
  $\left[\left(2S^{\mu}+\frac{Q^{\mu}}{2}-\tilde{Q}^{\mu}\right)u_{\lambda}-g^{\mu\nu}\nabla_{\nu}u_{\lambda}\right]\psi$
  $+u^{\mu}u_{\lambda}(\dot{\chi}+\dot{\psi})-(\phi+\chi+\psi+\omega)(\dot{u}^{\mu}u_{\lambda}+u^{\mu}\dot{u}_{\lambda})=0$

Pure Shear

Considering a pure shear hyperfluid (i.e. setting spin and dilation to zero) we have $\Phi = \frac{\kappa}{2}(\psi - \phi)$, $\zeta = 0$, $C = \frac{(n-1)}{2}B = -\kappa(n-1)\phi$, $A = \kappa(\phi - 2\psi)$ and the conservation laws become

$$\dot{\phi} + \Big((n-1) + 2w_1\Big)H\phi = 0$$

$$\dot{\rho} + (n-1)H(\rho + p) = 0$$

where we have assumed a shear 'equation of state' $\psi = w_1\phi$ with w_1 being a constant which we may refer to as the 'shear' barotropic index.

Note

The continuity equation completely decouples from the hypermomentum dof and has its usual form. This is not always true and is a particular feature of the **pure shear case.**

Friedmann Equations

After some algebra and upon using the conservation laws we finally obtain the simple expressions (also setting $n = 4$):

$$H^2 = \frac{\kappa}{3}\rho + \frac{\kappa^2 \phi^2}{4}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) - \frac{\kappa^2 \phi^2}{2}(1 + w_1)$$

- For $w_1 > -1$ shear hypermomentum slows down expansion
- For $w_1 < -1$ it accelerates the expansion and
- For $w_1 = -1$ has no effect on the acceleration

"Dusty Shear"

For $w_1 = 0$ from the conservation law of ϕ it follows that

$$\phi = \phi_0 \left(\frac{a_0}{a}\right)^3 = \frac{\lambda_0}{a^3}$$

and the Friedmann equations become

$$H^2 = \frac{\kappa}{3}\rho + \frac{\kappa^2\lambda_0^2}{4}\frac{1}{a^6}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) - \frac{\kappa^2\lambda_0^2}{2}\frac{1}{a^6}$$

Connection with stiff-matter

We see that the effect of the pure shear hyperfluid is indistinguishable from that of stiff matter.

• Stiff matter may not be so exotic and can be seen as a specific degree of freedom of the perfect hyperfluid!

Generalization: There exists a Perfect Hyperfluid, generalizing the Perfect Fluid notion of GR, for which: (D.I. 2021, JCAP)

$$t_{\mu\nu} = \tilde{\rho} u_\mu u_\nu + \tilde{p} h_{\mu\nu} \quad , \quad T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}$$

$$\Delta^{(n)}_{\alpha\mu\nu} = \phi h_{\mu\alpha} u_\nu + \chi h_{\nu\alpha} u_\mu + \psi u_\alpha h_{\mu\nu} + \omega u_\alpha u_\mu u_\nu + \delta_{n,4} \epsilon_{\alpha\mu\nu\kappa} u^\kappa \zeta$$

These sources are subject to the conservation laws:

$$\tilde{\nabla}_\mu t^\mu{}_\alpha = \frac{1}{2} \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha} + \frac{1}{2} Q_{\alpha\mu\nu} (t^{\mu\nu} - T^{\mu\nu})$$

$$t^\mu{}_\lambda = T^\mu{}_\lambda - \frac{1}{2\sqrt{-g}} \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda{}^{\mu\nu})$$

Remark

The Perfect Hyperfluid is a direct generalization of the Perfect Fluid description where now the microscopic characteristics of matter are also taken into account.

Conclusions/Further Prospects

- We have constructed the Perfect Cosmological Hyperfluid
- It can be further generalized by dropping the homogeneity assumption (Perfect Hyperfluid=Generalization of Perfect Fluid by taking into account the microstructure)
- The results apply also to Teleparallel Gravity (apart from MAG)
- Cosmological Solutions?
- Non-relativistic limit of hyperfluid?
- Connection to observations and bounds on hypermomentum variables?

...Thank you!!!