# Dynamics of Extended Bodies with Spin-Induced Quadrupole

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NEB-21, Corfu
1-4, September, 2025

Astronomický ústav AV ČR

# Motivation

- How can we define an extended body in a curved spacetime?
- How can we set a reference point in an extended body?
- Do different choices of a reference point describe the same physical body?
- How does the spin and quadrupole structure affect the motion of the body?

# Mathisson's gravitational skeleton

Recipe:

- Test body in curved spacetime.
- Multipole expansion of the stress-energy tensor of the body.
- Expansion truncation at a certain multipole:
  - zero order: mass monopole $P^{\mu}$,
  - first order: spin-dipole $S^{\mu\nu}$,
  - second order: quadrupole $J^{\mu\nu\kappa\lambda}$.
  - ...

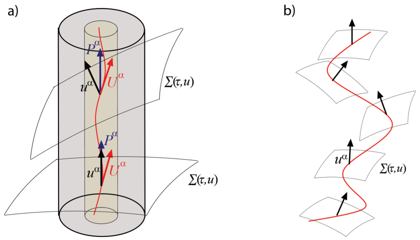


Figure: The worldtube of an extended body following the red worldline. $\Sigma(\tau, u)$ is the spacelike hypersurface generated by all geodesics orthogonal to the timelike vector $u^{\mu}$

• F. Costa, & J. Natario: *Equations of Motion in Relativistic Gravity*, pp 215-258 (2015). arXiv:2505.16783v2.

# Mathisson-Papapetrou-Dixon (MPD) equations

$$\frac{Dp^\mu}{d\tau} = 0 - \frac{1}{2} R^\mu_{\rho\beta\alpha} u^\rho S^{\beta\alpha} - \frac{1}{6} J^{\nu\rho\beta\alpha} \nabla^\mu R_{\nu\rho\beta\alpha},$$

$$\frac{DS^{\mu\nu}}{d\tau} = 2 p^{[\mu} u^{\nu]} + \frac{4}{3} R_{\alpha\beta\rho}{}^{[\mu} J^{\nu]\rho\alpha\beta}.$$

- Geodesic equation: momentum $p^\mu$
- Mathisson (1937), Papapetrou (1951): spin / dipole $S^{\mu\nu}$
- Dixon (∼1974): quadrupole $J^{\mu\nu\alpha\beta}$, ...
- EoM for $p^\alpha$ and $S^{\alpha\beta}$ follow from theory! $T^{\mu\nu}{}_{;\nu} = 0 \Rightarrow$ EoM
- Spin-supplementary condition (SSC): $S^{\mu\nu} V_\mu = 0,\ V^\mu V_\mu = -1$

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**Conserved Quantities:**

- For a Killing vector field $\xi^\mu$: $C(\xi) = p_\mu \xi^\mu + \frac{1}{2}\, S^{\mu\nu} \nabla_\mu \xi_\nu$
- Neglecting $J^{\mu\nu\alpha\beta}$ etc.: mass: $\mu = \sqrt{-p_\mu p^\mu}$ or $m = -u^\mu p_\mu$ (SSC dep.)

spin-length: $S = \sqrt{\frac{1}{2}\, S_{\mu\nu} S^{\mu\nu}}$

# Non-spinning & spinning test body motion

## Non-spinning

- Regular motion (integrable)
- Geodesic motion
- No spin–orbit coupling

## Spinning

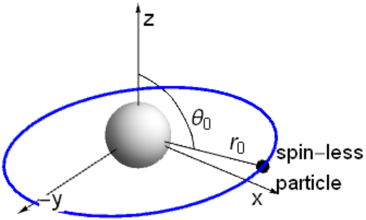
- Chaotic motion (non-integrable)
- Non-geodesic motion
- Spin–orbit coupling

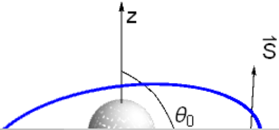
**Pole–dipole**

- $\mu$ conserved

**Pole-dipole-quadrupole**

- $\mu$ not conserved
- Extra spin$^2$/curvature couplings



# Spin-induced quadrupole and SSCs in Kerr background

## Spin-induced quadrupole moment

$$J^{\alpha\beta\gamma\delta} = -3V^{[\alpha}Q^{\beta][\gamma}V^{\delta]}, \qquad Q^{\alpha\beta} = k\, S^{\alpha}{}_{\gamma}\, S^{\beta\gamma}.$$

- for black holes: $k = 1$
- for neutron stars: $4 \lesssim k \lesssim 6$

## Fixing the center of mass

To fix the center of mass, we use two standard SSCs:

- Mathisson–Pirani (MP): $u_\mu S^{\mu\nu} = 0$
- Tulczyjew–Dixon (TD): $p_\mu S^{\mu\nu} = 0$


- M. Mathisson: *Acta Phys. Polonica*, **6**, 163 (1937).
- W. Tulczyjew: *Acta Phys. Pol.*, **18**, 393 (1959)

# Circular equatorial orbits (Kerr BH)

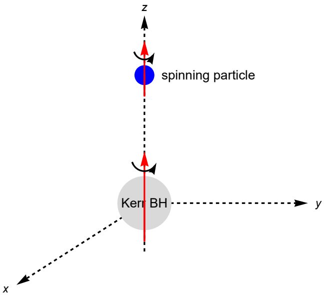
- Equatorial plane, i.e., $\theta = \frac{\pi}{2},\ p^\theta = 0$
- Circular orbits, i.e., $p^r = u^r = 0$
- $u^t \to u^\alpha u_\alpha = -1;\ p^t \to p^\alpha p_\alpha = -\mu^2$
- Aligned spin:

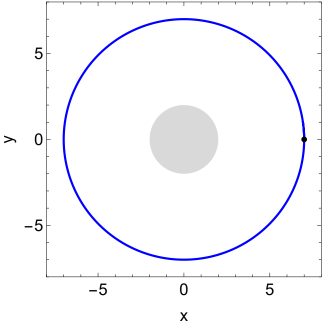
$$
\begin{aligned}
S^{tr} &= -S\sqrt{-\frac{g_{\theta\theta}}{g}}\ V_\phi = -S^{rt}, \\
S^{r\phi} &= -S\sqrt{-\frac{g_{\theta\theta}}{g}}\ V_t = -S^{\phi r}.
\end{aligned}
$$

- Non-trivial MPD Eqs:

$$
\frac{\mathrm{d}p^r}{\mathrm{d}\tau} = \frac{\mathrm{d}S^{t\phi}}{\mathrm{d}\tau} = 0.
$$

- Orbital frequency: $\Omega = \frac{u^\phi}{u^t} = \Omega_n \sigma^n + \mathcal{O}(\sigma^6)$

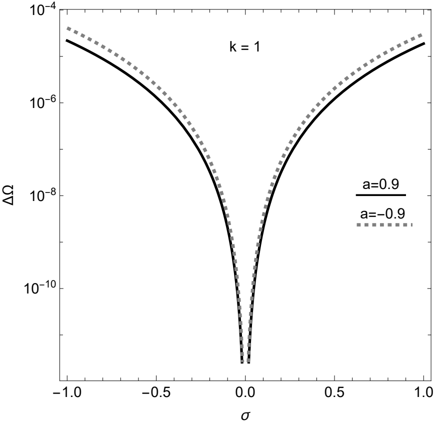


# Orbital frequency ($\Omega_\phi$) under TD and MP SSCs

| $\hat{\Omega}_n$ | TD SSC | MP SSC |
|---|---|---|
| $\mathcal{O}(\sigma^0)$ | $\frac{1}{\hat{a}\pm\sqrt{\hat{r}^3}}$ | $\frac{1}{\hat{a}\pm\sqrt{\hat{r}^3}}$ |
| $\mathcal{O}(\sigma^1)$ | $\frac{3\left(\pm\hat{a}-\sqrt{\hat{r}}\right)}{2\sqrt{\hat{r}}\left(\hat{a}\pm\sqrt{\hat{r}^3}\right)^2}$ | $\frac{3\left(\pm\hat{a}-\sqrt{\hat{r}}\right)}{2\sqrt{\hat{r}}\left(\hat{a}\pm\sqrt{\hat{r}^3}\right)^2}$ |
| $\mathcal{O}(\sigma^2)$ | $\hat{\Omega}_2(\hat{r},\hat{a},\hat{k})$ | $\hat{\Omega}_2(\hat{r},\hat{a},\hat{k})$ |
| $\mathcal{O}(\sigma^3)$ | $\hat{\Omega}_{3,\mathrm{TD}}(\hat{r},\hat{a},\hat{k})$ | $\hat{\Omega}_{3,\mathrm{MP}}(\hat{r},\hat{a},\hat{k})$ |
| $\mathcal{O}(\sigma^4)$ | $\hat{\Omega}_{4,\mathrm{TD}}(\hat{r},\hat{a},\hat{k})$ | $\hat{\Omega}_{4,\mathrm{MP}}(\hat{r},\hat{a},\hat{k})$ |
| $\mathcal{O}(\sigma^5)$ | $\hat{\Omega}_{5,\mathrm{TD}}(\hat{r},\hat{a},\hat{k})$ | $\hat{\Omega}_{5,\mathrm{MP}}(\hat{r},\hat{a},\hat{k})$ |
| $\mathcal{O}(\sigma^6)$ | $\hat{\Omega}_{6,\mathrm{TD}}(\hat{r},\hat{a},\hat{k})$ | $\hat{\Omega}_{6,\mathrm{MP}}(\hat{r},\hat{a},\hat{k})$ |

Table: Power series expansion coefficients for orbital frequency $\hat{\Omega}_\pm$ of spinning test body with spin-induced quadrupole.



# Centroid shift (radial shift)

Shifting from one centroid $z^\alpha$ to another centroid $\tilde{z}^\alpha$:

- Radial shift: $\delta z^\nu = \frac{\tilde{p}_\mu S^{\mu\nu}}{\tilde{\mu}^2}$.

For $\hat{k} = 1$:

- $\hat{\Omega}'_{4,\mathrm{TD}} = \hat{\Omega}_{4,\mathrm{MP}}$,
- $\hat{\Omega}'_{5,\mathrm{TD}} = \hat{\Omega}_{5,\mathrm{MP}}$,
- $\hat{\Omega}'_{6,\mathrm{TD}} \neq \hat{\Omega}_{6,\mathrm{MP}}$.

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| $\mathcal{O}(\sigma^4)$ | $\hat{\Omega}'_{4,\mathrm{TD}}(\hat{r}, \hat{a}, \hat{k})$ | $\hat{\Omega}_{4,\mathrm{MP}}(\hat{r}, \hat{a}, \hat{k})$ |
| $\mathcal{O}(\sigma^5)$ | $\hat{\Omega}'_{5,\mathrm{TD}}(\hat{r}, \hat{a}, \hat{k})$ | $\hat{\Omega}_{5,\mathrm{MP}}(\hat{r}, \hat{a}, \hat{k})$ |
| $\mathcal{O}(\sigma^6)$ | $\hat{\Omega}'_{6,\mathrm{TD}}(\hat{r}, \hat{a}, \hat{k})$ | $\hat{\Omega}_{6,\mathrm{MP}}(\hat{r}, \hat{a}, \hat{k})$ |

Table: Power series expansion coefficients for orbital frequency $\hat{\hat{\Omega}}_\pm$ (after radial shifting).

# Centroid shift (both radial and spin shifts)

Shifting from one centroid $z^{\alpha}$ to another centroid $\tilde{z}^{\alpha}$:

- Radial shift: $\delta z^{\nu} = \frac{\tilde{p}_{\mu} S^{\mu\nu}}{\tilde{\mu}^2}$
- Spin-shift: $\tilde{S}^{\mu\nu} = S^{\mu\nu} + p^{\mu}\delta z^{\nu} - p^{\nu}\delta z^{\mu}$.

For $\hat{k} = 1$:

- $\hat{\Omega}''_{4,\mathrm{TD}} = \hat{\Omega}_{4,\mathrm{MP}}$,
- $\hat{\Omega}''_{5,\mathrm{TD}} \neq \hat{\Omega}_{5,\mathrm{MP}}$.

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Table: Power series expansion coefficients for orbital frequency $\hat{\hat{\Omega}}_{\pm}$ of spinning test body (after both radial and spin shifts).

# Summary & Future work

- The orbital frequencies under the TD and MP SSCs are equivalent up to the quadratic order in spin $\sigma$, i.e., the terms up to $\mathcal{O}(\sigma^2)$ are identical.
- When the centroid position corrections are applied, for the case of BHs ($\hat{k} = 1$), the orbital frequencies under the TD and MP SSCs are equivalent up to order five in spin $\sigma$.
- When the spin corrections are applied, for the case of BHs, the orbital frequencies under the TD and MP SSCs are equivalent up to order four in spin $\sigma$.

- We are working on modelling the gravitational waveforms from a spinning secondary with spin-induced quadrupole.

Thank you for your attention.

• M. Shahzadi, G. Lukes-Gerakopoulos, M. Kološ: *Circular equatorial orbits of extended bodies with spin-induced quadrupole around a Kerr black hole: Comparing spin-supplementary conditions*, Phys. Rev. D; arXiv:2505.16783v2.