Dynamics of Extended Bodies with Spin-Induced Quadrupole

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Motivation

- How can we define an extended body in a curved spacetime?
- How can we set a reference point in an extended body?
- Do different choices of a reference point describe the same physical body?
- How does the spin and quadrupole structure affect the motion of the body?

Mathisson's gravitational skeleton

Recipe:

- Test body in curved spacetime.
- Multipole expansion of the stress-energy tensor of the body.
- Expansion truncation at a certain multipole:
  - zero order: mass monopole P^{μ},
  - first order: spin-dipole $S^{\mu\nu}$,
  - second order: quadrupole $J^{\mu\nu\kappa\lambda}$.
  - ...

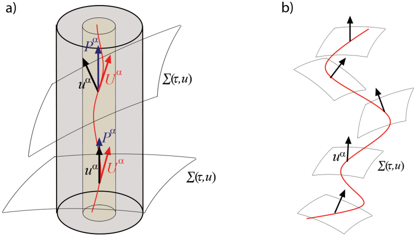

Figure: The worldtube of an extended body following the red worldline. $\Sigma(\tau, u)$ is the spacelike hypersurface generated by all geodesics orthogonal to the timelike vector u^{μ}

• F. Costa, & J. Natario: *Equations of Motion in Relativistic Gravity*, pp 215-258 (2015). arXiv:2505.16783v2.

Mathisson-Papapetrou-Dixon (MPD) equations

$$\frac{\mathrm{D}p^{\mu}}{\mathrm{d}\tau} = 0 - \frac{1}{2} R^{\mu}_{\ \rho\beta\alpha}\, u^{\rho}\, S^{\beta\alpha} - \frac{1}{6} J^{\nu\rho\beta\alpha}\, \nabla^{\mu} R_{\nu\rho\beta\alpha},$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2\, p^{[\mu} u^{\nu]} + \frac{4}{3} R_{\alpha\beta\rho}{}^{[\mu}\, J^{\nu]\rho\alpha\beta}.$$

- Geodesic equation: momentum p^{μ}
- Mathisson (1937), Papapetrou (1951): spin / dipole $S^{\mu\nu}$
- Dixon ($\sim$1974): quadrupole $J^{\mu\nu\alpha\beta}$, ...
- EoM for p^{α} and $S^{\alpha\beta}$ follow from theory! $T^{\mu\nu}{}_{;\nu} = 0 \Rightarrow$ EoM
- Spin-supplementary condition (SSC): $S^{\mu\nu}V_{\mu} = 0,\ V^{\mu}V_{\mu} = -1$

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Conserved Quantities:

- For a Killing vector field ξ^{μ}: $C(\xi) = p_{\mu}\xi^{\mu} + \frac{1}{2} S^{\mu\nu}\nabla_{\mu}\xi_{\nu}$
- Neglecting $J^{\mu\nu\alpha\beta}$ etc.: mass: $\mu = \sqrt{-p_{\mu}p^{\mu}}$ or $m = -u^{\mu}p_{\mu}$ (SSC dep.)

spin-length: $S = \sqrt{\frac{1}{2} S_{\mu\nu} S^{\mu\nu}}$

Non-spinning & spinning test body motion

Non-spinning

- Regular motion
- Integrable
- Geodesic motion
- No spin–orbit coupling

Spinning

- Chaotic motion
- Non-integrable
- Non-geodesic motion
- Spin–orbit coupling

Pole–dipole

- μ conserved

Pole-dipole-quadrupole

- μ not conserved
- Extra spin^2/curvature couplings

Spin-induced quadrupole and SSCs in Kerr background

Spin-induced quadrupole moment

$$J^{\alpha\beta\gamma\delta} = -3V^{[\alpha}Q^{\beta][\gamma}V^{\delta]}, \qquad Q^{\alpha\beta} = k\,S^{\alpha}{}_{\gamma}\,S^{\beta\gamma}.$$

- For black holes: $k = 1$
- For neutron stars: $4 \lesssim k \lesssim 6$

Fixing the center of mass

To fix the center of mass, we use two standard SSCs:

- Mathisson–Pirani (MP): $u_\mu S^{\mu\nu} = 0$
- Tulczyjew–Dixon (TD): $p_\mu S^{\mu\nu} = 0$

- M. Mathisson: *Acta Phys. Polonica*, **6**, 163 (1937).
- W. Tulczyjew: *Acta Phys. Pol.*, **18**, 393 (1959)

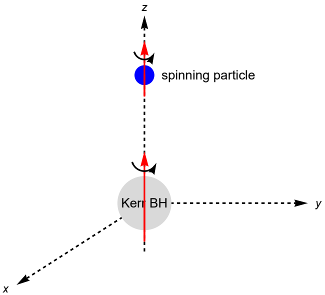
Circular equatorial orbits (Kerr BH)

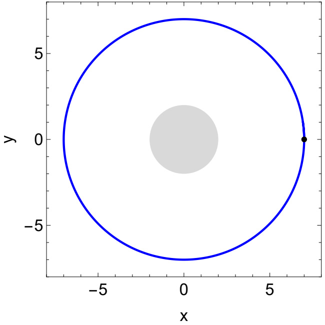
- Equatorial plane, i.e., $\theta = \frac{\pi}{2},\ p^\theta = 0$
- Circular orbits, i.e., $p^r = u^r = 0$
- $u^t \to u^\alpha u_\alpha = -1;\ p^t \to p^\alpha p_\alpha = -\mu^2$
- Aligned spin:

$$S^{tr} = -S\sqrt{-\frac{g_{\theta\theta}}{g}}\ V_\phi = -S^{rt},$$
$$S^{r\phi} = -S\sqrt{-\frac{g_{\theta\theta}}{g}}\ V_t = -S^{\phi r}.$$

- Non-trivial MPD Eqs: $\frac{\mathrm{d}p^r}{\mathrm{d}\tau} = \frac{\mathrm{d}S^{t\phi}}{\mathrm{d}\tau} = 0.$
- Orbital frequency:

$$\Omega = \frac{u^\phi}{u^t} = \Omega_n \sigma^n + \mathcal{O}(\sigma^7), \quad n = 0, \cdots, 6.$$

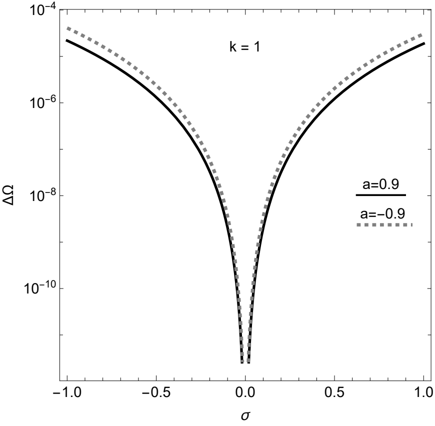




Orbital frequency (Ω_ϕ) under TD and MP SSCs

$\hat{\Omega}_n$	TD SSC	MP SSC
$\mathcal{O}(\sigma^0)$	$\frac{1}{\hat{a}\pm\sqrt{\hat{r}^3}}$	$\frac{1}{\hat{a}\pm\sqrt{\hat{r}^3}}$
$\mathcal{O}(\sigma^1)$	$\frac{3\left(\pm\hat{a}-\sqrt{\hat{r}}\right)}{2\sqrt{\hat{r}}\left(\hat{a}\pm\sqrt{\hat{r}^3}\right)^2}$	$\frac{3\left(\pm\hat{a}-\sqrt{\hat{r}}\right)}{2\sqrt{\hat{r}}\left(\hat{a}\pm\sqrt{\hat{r}^3}\right)^2}$
$\mathcal{O}(\sigma^2)$	$\hat{\Omega}_2(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_2(\hat{r}, \hat{a}, \hat{k})$
$\mathcal{O}(\sigma^3)$	$\hat{\Omega}_{3,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{3,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$
$\mathcal{O}(\sigma^4)$	$\hat{\Omega}_{4,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{4,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$
$\mathcal{O}(\sigma^5)$	$\hat{\Omega}_{5,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{5,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$
$\mathcal{O}(\sigma^6)$	$\hat{\Omega}_{6,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{6,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$

Table: Power series expansion coefficients for orbital frequency $\hat{\Omega}_\pm$ of spinning test body with spin-induced quadrupole.



Centroid shift (radial shift)

Shifting from one centroid z^α to another centroid $\tilde{z}^\alpha$:

- Radial shift: $\delta z^\nu = \frac{\tilde{p}_\mu S^{\mu\nu}}{\tilde{\mu}^2}$.

For $\hat{k} = 1$:

- $\hat{\Omega}'_{4,\text{TD}} = \hat{\Omega}_{4,\text{MP}}$,
- $\hat{\Omega}'_{5,\text{TD}} = \hat{\Omega}_{5,\text{MP}}$,
- $\hat{\Omega}'_{6,\text{TD}} \neq \hat{\Omega}_{6,\text{MP}}$.

$\hat{\Omega}_n$	TD SSC	MP SSC
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Table: Power series expansion coefficients for orbital frequency $\hat{\hat{\Omega}}_\pm$ (after radial shifting).

Centroid shift (both radial and spin shifts)

Shifting from one centroid z^α to another centroid $\tilde{z}^\alpha$:

- Radial shift: $\delta z^\nu = \frac{\tilde{p}_\mu S^{\mu\nu}}{\tilde{\mu}^2}$
- Spin-shift: $\tilde{S}^{\mu\nu} = S^{\mu\nu} + p^\mu \delta z^\nu - p^\nu \delta z^\mu$.

For $\hat{k} = 1$:

- $\hat{\Omega}''_{4,\text{TD}} = \hat{\Omega}_{4,\text{MP}}$,
- $\hat{\Omega}''_{5,\text{TD}} \neq \hat{\Omega}_{5,\text{MP}}$.

$\hat{\Omega}_n$	TD SSC	MP SSC
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$\mathcal{O}(\sigma^4)$	$\hat{\Omega}''_{4,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{4,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$
$\mathcal{O}(\sigma^5)$	$\hat{\Omega}''_{5,\text{TD}}(\hat{r}, \hat{a}, \hat{k})$	$\hat{\Omega}_{5,\text{MP}}(\hat{r}, \hat{a}, \hat{k})$

Table: Power series expansion coefficients for orbital frequency $\hat{\hat{\Omega}}_\pm$ of spinning test body (after both radial and spin shifts).

Summary & Future work

- The orbital frequencies under the TD and MP SSCs are equivalent up to the quadratic order in spin σ, i.e., the terms up to $\mathcal{O}(\sigma^2)$ are identical.
- When the centroid position corrections are applied, for the case of BHs ($\hat{k} = 1$), the orbital frequencies under the TD and MP SSCs are equivalent up to order five in spin σ.
- When the spin corrections are applied, for the case of BHs, the orbital frequencies under the TD and MP SSCs are equivalent up to order four in spin σ.

- We are working on modelling the gravitational waveforms from a spinning secondary with spin-induced quadrupole.

Thank you for your attention.

• M. Shahzadi, G. Lukes-Gerakopoulos, M. Kološ: *Circular equatorial orbits of extended bodies with spin-induced quadrupole around a Kerr black hole: Comparing spin-supplementary conditions*, Phys. Rev. D; arXiv:2505.16783v2.