Penrose inequality for integral energy conditions

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NEB21
1 September 2025

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Outline

Based on 2504.19794 with Eduardo Hafemann.

Introduction and motivation

Classical relativity theorems

- Singularity theorems: prove geodesic incompleteness [Penrose, 1964], [Hawking, 1966], [Hawking, Penrose, 1970]
- Black hole area theorem: proves that the black hole event horizon can never decrease [Hawking, 1971]
- Topology theorem: proves that cross sections of the event horizon for asymptotically flat black holes are topologically 2-spheres [Hawking, 1971]
- Penrose inequality: relation between the ADM and the area of any cross section of the black hole event horizon [Penrose, 1973]

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All these theorems use classical energy conditions in their assumptions.

Energy conditions

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Name	Physical	Geometric	Perfect fluid
DEC	$T_{\mu\nu} t^\mu \xi^\nu \geq 0$	$G_{\mu\nu} t^\mu \xi^\nu \geq 0$	$\rho \geq \lvert P \rvert$
NEC	$T_{\mu\nu} U^\mu U^\nu \geq 0$	$R_{\mu\nu} U^\mu U^\nu \geq 0$	$\rho + P \geq 0$

t^μ and ξ^μ: co-oriented timelike vectors, U^μ: null vector

Semiclassical gravity

Question

Do these theorems generalize to semiclassical gravity?

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Quantum fields generating a classical curved background:

$$8\pi G\langle :T_{\mu\nu}:\rangle_\psi = G_{\mu\nu}$$

- Approximate equation with certain regime of validity
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- Very few cases where we actually know solutions exist

Positivity of the energy density is in general incompatible with quantum field theory. There is a nonzero probability for both positive and negative measurement values, so the spectrum of observables extends into the negative half-line. All pointwise conditions are violated in that context. [Epstein, Glaser, Jaffe, 1965]

Semiclassical theorems

General strategy:

- Step 1: Replace the pointwise condition by an average one and prove the theorem

$$\int_\gamma f^2 R_{\mu\nu} U^\mu U^\nu \geq -(\text{bound})$$

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- Step 3: Use the semiclassical Einstein equation

$$8\pi G \langle :T_{\mu\nu}: U^\mu U^\nu \rangle_\psi = R_{\mu\nu} U^\mu U^\nu$$

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QFT

Quantum energy inequalities (QEIs) introduce a restriction on the possible magnitude and duration of any negative energy densities within a quantum field theory.

Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle : T_{\mu\nu} t^\mu t^\nu : \rangle_\psi \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

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Weakened condition inspired by QEIs

$$\int f^2 R_{\mu\nu} U^\mu U^\nu \, dt \geq -Q_m \| f^{(m)} \|^2 - Q_0 \| f \|^2$$

The original Penrose inequality

The Penrose argument

- Everywhere on M and for all null vectors U^μ

$$R_{\mu\nu} U^\mu U^\nu \geq 0$$

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- M contains a trapped surface T so a singularity forms
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- The area theorem holds so the event horizon of a black hole is always non-decreasing
- The black hole spacetime settles to a Kerr black hole so

$$A \leq A_{\text{Kerr}} \, ,$$

The Penrose argument

The Bondi mass

The mass of a gravitating system defined in terms of the asymptotic behavior at *null infinity*.

The ADM mass

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$$A_{\mathrm{Kerr}} = 8\pi M_{\mathrm{Kerr}} \left(M_{\mathrm{Kerr}} + \sqrt{M_{\mathrm{Kerr}}^2 - a^2} \right) \leq 16\pi M_{\mathrm{Kerr}}^2 \,,$$

$$16\pi M_{\mathrm{Kerr}}^2 = 16\pi M_{\mathrm{Bondi}}^2 \leq 16\pi m_{\mathrm{ADM}}^2 \,.$$

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$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A}{16\pi}}$$

The initial data formulation

Use properties of an initial data set instead of the whole spacetime.

- Consider a time-orientable spacetime $(\overline{M}^n, \overline{g})$ satisfying the Einstein equations and *initial data set* $(M^d, g, \mathcal{K})$
- The DEC holding in $\bar{M}$ guarantees that

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Theorem

Let $(M, g, \mathcal{K})$, be a complete spherically symmetric, asymptotically flat initial data set such that ∂M is an outermost marginally outer trapped surface (MOTS). If the DEC holds

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{|\partial M|}{\omega_{d-1}}\right)^{\frac{d-2}{d-1}}.$$

Penrose inequality for evaporating black holes

The generalizations

Two generalizations

1. Initial data formulation of the Penrose inequality with an average DEC
2. Using the previous formulation and a condition inspired by QEIs to acquire a Penrose inequality for evaporating black holes

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Theorem [Hafemann, E-AK, 2025]

Let $(M, g, \mathcal{K})$, be a complete spherically symmetric, asymptotically flat initial data set such that ∂M is an outermost MOTS. If the average dominant energy condition (ADEC) holds

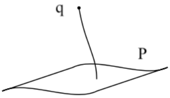
$$\int_{r_0}^{\infty} r^{n-1} \left(\mu - |J|_g\right) \, dr \geq 0,$$

then

$$m_{\mathrm{ADM}} \geq \frac{1}{2} \left(\frac{|\partial M|}{\omega_{d-1}} \right)^{\frac{d-2}{d-1}} .$$

Focusing theorem

A focal point on γ is a point where a causal geodesic no longer extremizes the action integral.



Energy of action integral

$$E[\gamma] = \frac{1}{2}\int_0^{\ell} g(\gamma'(\lambda), \gamma'(\lambda))d\lambda$$

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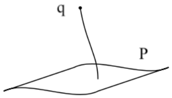

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The Hessian

$$\mathbf{H}[V] \equiv \frac{d^2E[\gamma_s]}{ds^2}\Big|_{s=0} =$$

$$\int_0^\ell \left[(\nabla_U V_\mu)(\nabla_U V^\mu) + R_{\mu\nu\alpha\beta}\overbrace{U^\mu}^{\text{tangent}}\underbrace{V^\nu}_{\text{variation}}V^\alpha U^\beta\right]d\lambda - U_\mu \mathrm{II}^\mu(V,V)\Big|_{\gamma(0)}$$

Focusing theorem

Let e_i with $i = 1, \dots, n-2$ be an orthonormal basis of $T_{\gamma(0)}P$, and parallel transport them along γ to generate $\{E_i\}_{i=1,\dots,n-2}$. Then, take f a smooth function with $f(0) = 1$ and $f(\ell) = 0$ and sum over i

$$\sum_{i=1}^{n-2} \mathbf{H}(fE_i, fE_i) = -\int_0^{\ell} \left((n-2) f'^2(\lambda) - f^2 R_{\mu\nu} U^\mu U^\nu \right) d\lambda - (n-2) f^2 U_\mu \mathrm{H}^\mu \Big|_{\gamma(0)}$$

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Condition for the existence of focal points

- No focal point before q: $\mathbf{H}[V] > 0$ for all V
- Focal point before q: $\mathbf{H}[V] < 0$ for some V
- $\mathbf{H}[V] = 0$ marginal case

The sufficiently trapped surface

Condition for the formation of a focal point

$$U_\mu H^\mu\big|_{\gamma(0)} \leq -J_\ell[f] = -\frac{1}{n-2}\int_0^\ell \left((n-2)f'(\lambda)^2 - f(\lambda)^2 R_{\mu\nu}U^\mu U^\nu\right)d\lambda$$

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Singularity theorem

Geodesic incompleteness for sufficiently trapped surface:

$$U_\mu H^\mu\big|_{\gamma(0)} \leq -\nu \equiv -\inf_f J_\ell[f]$$

[Fewster, E-AK, 2019]

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Area theorem

The sufficiently trapped surface is always inside the event horizon. [E-AK, Sacchi, 2023]

The sufficiently trapped surface

Penrose inequality for evaporating black holes

Theorem [Hafemann, E-AK, 2025]

(i) $(\overline{M}^{n+1}, \overline{g})$ is a spherically symmetric, strongly asymptotically predictable with a sufficiently trapped surface T.

(ii) Let M_1 and M_2 be asymptotically flat partial Cauchy surfaces for the globally hyperbolic region such that $M_1 \subset I^-(M_2)$.

Let $\mathcal{H}_0 \equiv \mathcal{H} \cap M_1$ and $\mathcal{H}_s \equiv \mathcal{H} \cap M_2$. If M_2 is an initial data set satisfying the ADEC and then

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{A_{\min}(T)}{\omega_{n-2}}\right)^{\frac{n-3}{n-2}} \exp\left(-\frac{1}{2(n-2)}\int_0^s \nu(\lambda)d\lambda\right).$$

where $A_{\text{min}}(T)$ is the minimum area required to enclose the sufficiently trapped surface T.

Application

We can replace the "sufficiently trapped surface" with

$$\int_0^\ell f(\lambda)^2 \overbrace{R_{\mu\nu} U^\mu U^\nu}^{\rho} \, d\lambda \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2$$

and $\rho \geq \rho_0$ for $[0, \ell_0]$. Then we can compute $\nu(\ell, \ell_0, Q_m, Q_0, \rho_0)$.
For $m = 1$ and taking $\ell \to \infty$

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A_{\min}(T)}{16\pi}} \exp\left(-\frac{1}{4}\nu_{\mathrm{opt}} \cdot s\right) ,$$

where $\nu_{\mathrm{opt}} = \nu_{\mathrm{opt}}(Q_0, Q_1, \rho_0)$

> The more negative energy allowed and the further apart the initial data sets lie, the more this inequality differs from the original one.

The smeared null energy condition

Smeared null energy condition (SNEC) for the minimally coupled scalar field in four dimensional Minkowski spacetime [Freivogel, Krommydas, 2018]

$$\int d\lambda \langle :T_{\mu\nu}: U^{\mu} U^{\nu} \rangle_{\psi} f^2(\lambda) \geq -\frac{4B}{G_N} \|f'\|^2$$

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What is B?

Numerical constant that expresses the scale of the UV cutoff

- $\ell_{\mathrm{UV}} \approx$ Planck length $\rightarrow B$ order 1 $\rightarrow$ A lot of negative energy allowed
- $\ell_{\mathrm{UV}} \gg$ Planck length $\rightarrow B$ small $\rightarrow$ A little negative energy allowed

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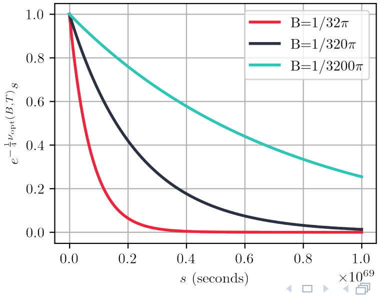
Semiclassical Einstein equation

$$8\pi G_N \langle :T_{\mu\nu}: U^\mu U^\nu \rangle_\psi = R_{\mu\nu} U^\mu U^\nu$$



Interpretation

'Global' form of inequalities

$$m_{\mathrm{ADM}} \geq \frac{1}{2}\sqrt{\frac{|\mathcal{H} \cap M_0|}{16\pi}} \exp\left(-\frac{1}{4}\nu_{\mathrm{opt}} \cdot s\right) , \qquad m_{\mathrm{ADM}} \geq \sqrt{\frac{A}{16\pi}}$$

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Problem: the event horizon is unknown before the full evolution.

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A_{\min}(T)}{16\pi}} \exp\left(-\frac{1}{4}\nu_{\mathrm{opt}} \cdot s\right),$$

Use: if the inequality is not satisfied for a given QEI, then the failure can be attributed to the non-formation of the event horizon.

Conclusions

- Proved versions of the Penrose inequality with average energy conditions
- Is there a relation between QEIs and the SDEC? Could it be applicable for quantum fields?
- Relationship with positive mass theorem?