Vector induced gravitational waves (VIGWs) sourced by primordial magnetic fields (PMFs)

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Scuola Superiore Meridionale

Talk Outline

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- Recap of the first order vector metric perturbations

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- PMFs as a source of vector metric perturbations

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- PMF-sourced VIGW spectrum
- GW detectability
- Conclusions - Outlook

First order vector perturbations

SVT Decomposition

$\Longrightarrow$ FLRW metric with scalar, vector and tensor perturbations

$$\mathrm{d}s^2 = a(\eta)^2\left[-(1+2\Phi)\,\mathrm{d}\eta^2 - 2\zeta_i \mathrm{d}\eta \mathrm{d}x^i + \left\{\left(1-2\Psi\right)\delta_{ij} + \left(\partial_i\xi_j + \partial_j\xi_i\right) + \frac{1}{2}h_{ij}\right\}\mathrm{d}x^i\mathrm{d}x^j\right]$$

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Gauge redudancy $\Longrightarrow$ Define a gauge-invariant variable $V_i = \zeta_i + \xi_i'$ with freedom to fix $\xi_i' = 0$.

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$$V_i' + 2\mathcal{H}V_i = 0 \Longrightarrow V_i \propto a^{-2}$$

V_i decay fastly in the absence of source.

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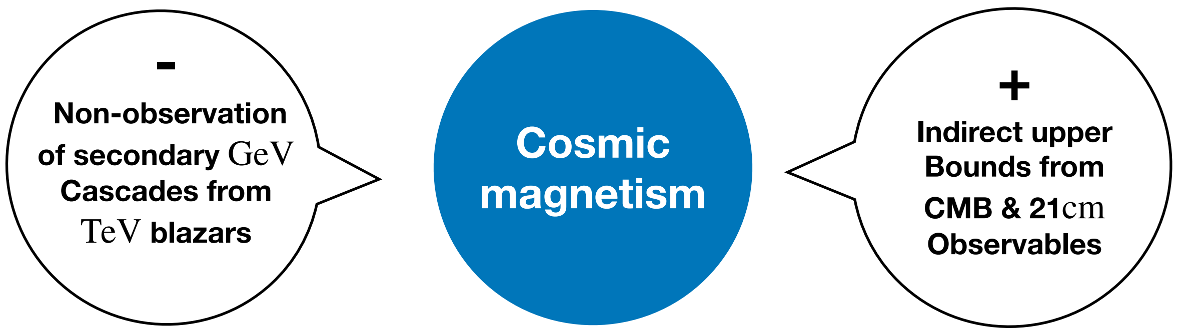
Primordial magnetic fields can generate such vector perturbations!

PMFs as a vector source



- **Galactic scales** $\Longrightarrow B^{(10\text{kpc})} \sim \mathcal{O}(10^{-6}\text{G})$
- **IGM scales** $\Longrightarrow \mathcal{O}(10^{-16}\text{G}) \lesssim B^{(1\text{Mpc})} \lesssim \mathcal{O}(10^{-11}\text{G})$

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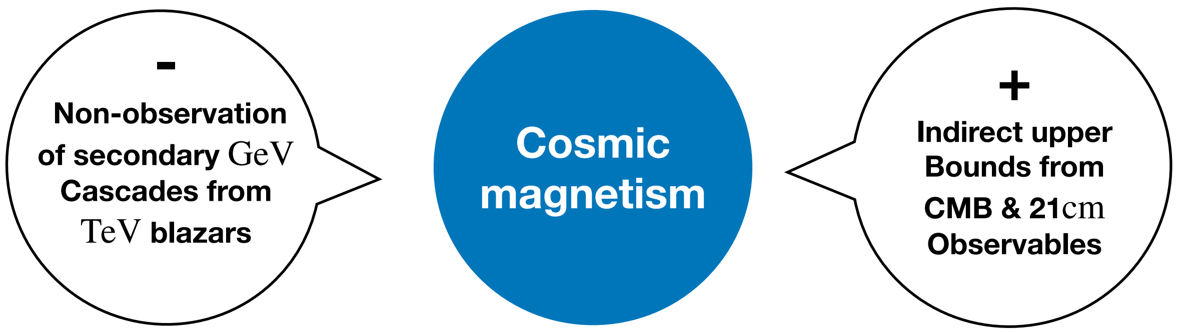

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Dynamo models face challenges at IGM scales

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Primordial origin ?

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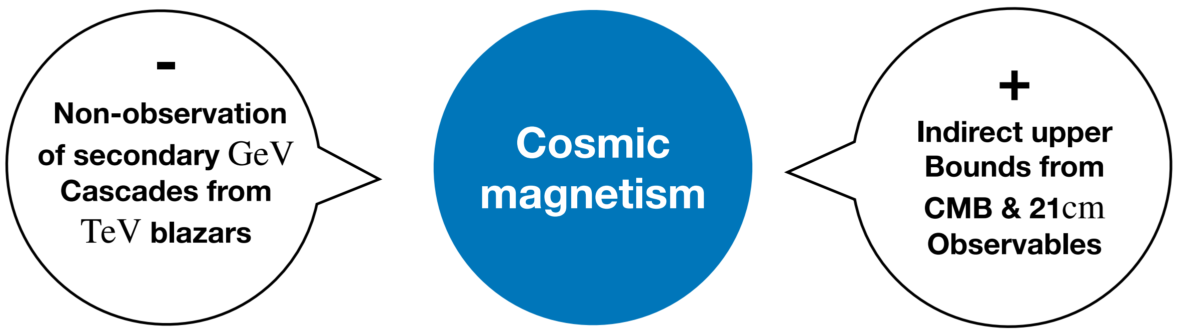

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Non-helical stochastic PMF: $\langle B_i(\boldsymbol{k})B_j(\boldsymbol{k}')\rangle = (2\pi)^3 P_{ij} P_B(k)\delta(\boldsymbol{k}+\boldsymbol{k}')$

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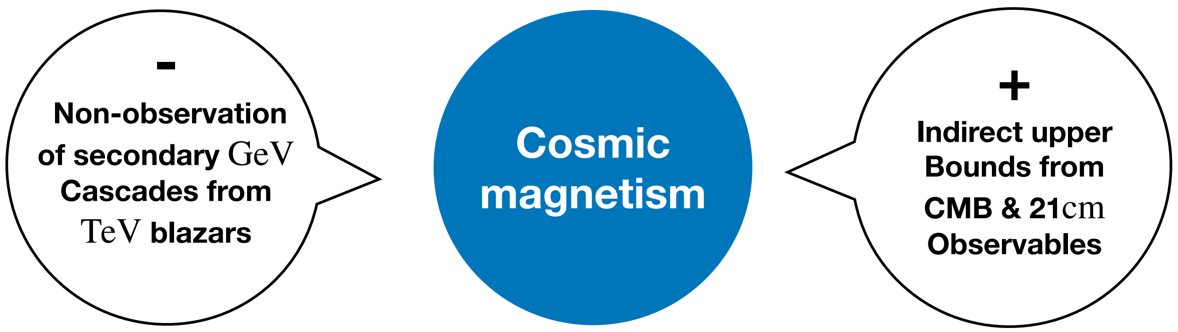

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Vector projected component of the coming PMF stress-energy tensor

$$\tau_{ij}^{(B)}(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \frac{1}{4\pi} \int \mathrm{d}^3\boldsymbol{p} \left[B_i(\boldsymbol{p}) B_j(\boldsymbol{k} - \boldsymbol{p}) - \frac{1}{2} \delta_{ij} B_l(\boldsymbol{p}) B_l(\boldsymbol{k} - \boldsymbol{p}) \right]$$

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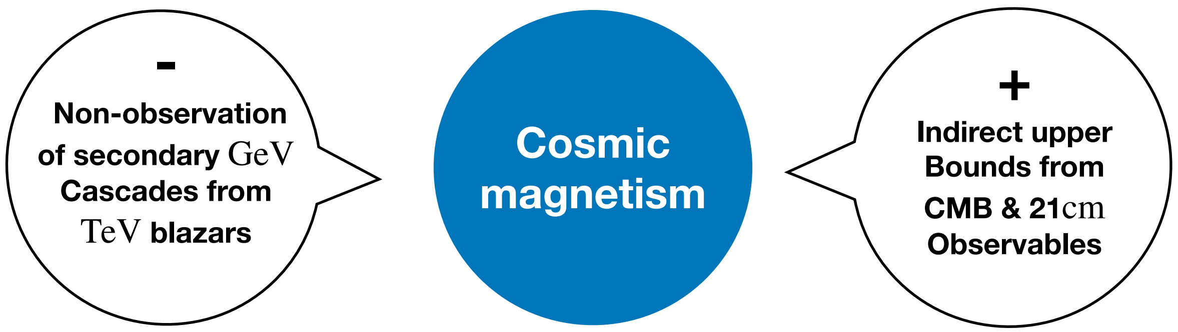

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[Mack et al. - 2001]

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$a(\eta) \propto \eta^{2/(1+3w)} \Rightarrow$ Time evolution of V_i depends crucially on the background EoS.

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Extreme scenario: $V_i \propto a^0$ for kination, i.e. $w = 1$.

Induced Gravitational Waves

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1

Inherent nonlinearity

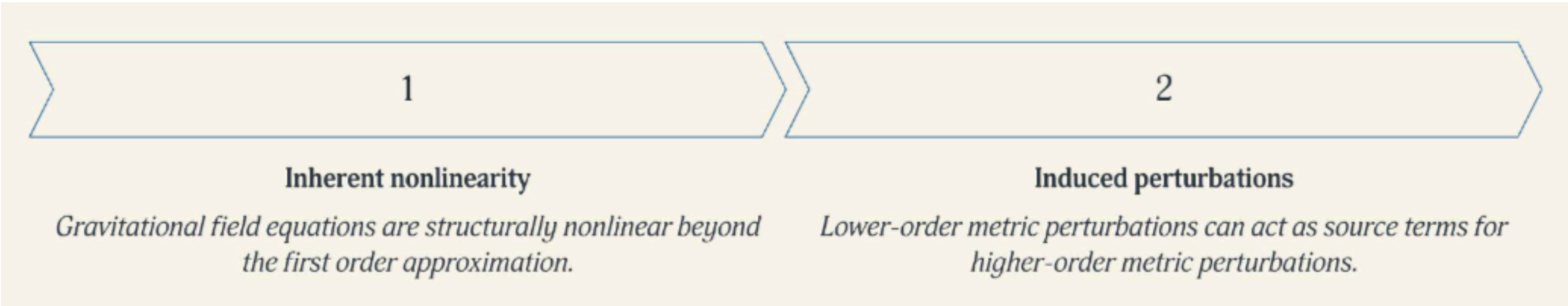
Gravitational field equations are structurally nonlinear beyond the first order approximation.

2

Induced perturbations

Lower-order metric perturbations can act as source terms for higher-order metric perturbations.

Induced Gravitational Waves



A few well-studied scenarios

- Scalar-induced gravitational waves (SIGWs) ⇒ Ananda et al (2007), Baumann et al (2007), Domenech (2021), etc.
- Tensor-induced scalar perturbations ⇒Bari et al (2022 & 2023).
- IGWs from scalar-tensor coupling ⇒Picard & Malik (2023), Bari et al (2023).
- Scalar-induced vector perturbations ⇒Mollerach et al (2003), Saga et al.(2015).
- Tensor-induced tensor perturbations ⇒Picard & Malik (2023), Gorji & Sasaki (2023).

Vector induced gravitational waves (VIGWs)

Primordial magnetic field

Carries spatial stress-energy $\tau_{ij}^{(B)}$

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Induced by first-order-vectors

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<u>Generic bilinear VIGW source function (using xPand)</u>

$$h_{ij}''(\boldsymbol{x},\eta) + 2\mathscr{H}(\eta)h_{ij}'(\boldsymbol{x},\eta) - \Delta h_{ij}(\boldsymbol{x},\eta) = \widehat{\mathscr{T}}_{ij}^{\,ab} S_{ab}(\boldsymbol{x},\eta)\,,$$

$$S_{ab}(\boldsymbol{x},\eta) = -V_c\partial_c(\partial_a V_b + \partial_b V_a) + \partial_c V_a \partial_c V_b + \partial_a V_c \partial_b V_c + 2V_c\partial_a\partial_b V_c + \frac{\Delta V_a \Delta V_b}{6\mathscr{H}^2(1+w)}$$

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Transforming to Fourier space + decomposing in helicity basis:

$$h_{\pm}''(\boldsymbol{k},\eta) + 2\mathscr{H}(\eta)h_{\pm}'(\boldsymbol{k},\eta) + k^2 h_{\pm}(\boldsymbol{k},\eta) = S_{\pm}(\boldsymbol{k},\eta)\,,$$

$$S_\lambda(\boldsymbol{k},\eta) = \frac{1}{\sqrt{2}}\int\frac{d^3q}{(2\pi)^{3/2}}\Bigg[e^{\ell}_{-\lambda}(\boldsymbol{k})V_\ell(\boldsymbol{q},\eta)e^{m}_{-\lambda}(\boldsymbol{k})V_m(\boldsymbol{k}-\boldsymbol{q},\eta)\left(\boldsymbol{q}\,.\,\left(\boldsymbol{q}-\boldsymbol{k}\right) + \frac{q^2|\boldsymbol{q}-\boldsymbol{k}|^2}{6(1+w)\mathscr{H}^2}\right)$$
$$-\left(e^{\ell}_{-\lambda}(\boldsymbol{k})q_\ell\right)^2 V_c(\boldsymbol{q},\eta_1)V_c(\boldsymbol{k}-\boldsymbol{q},\eta) - 2k_a V_a(\boldsymbol{q},\eta)e^{\ell}_{-\lambda}(\boldsymbol{k})q_\ell e^{m}_{-\lambda}(\boldsymbol{k})V_m(\boldsymbol{k}-\boldsymbol{q},\eta)\Bigg]$$

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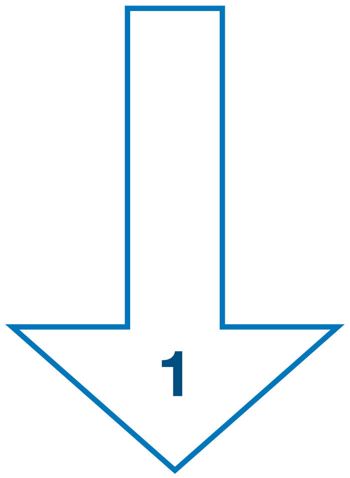

We need to compute the VIGW two-point correlator

$$\langle h_\lambda(\boldsymbol{k}_1,\eta)h_{\lambda'}(\boldsymbol{k}_2,\eta)\rangle = \int_{\eta_i}^{\eta} d\eta_1 \int_{\eta_i}^{\eta} d\eta_2\, g_{k_1}(\eta,\eta_1)g_{k_2}(\eta,\eta_2)\langle S_\lambda(\boldsymbol{k}_1,\eta_1)S_{\lambda'}(\boldsymbol{k}_2,\eta_2)\rangle,$$

where $g_k(\eta,\tilde{\eta}) = \dfrac{\pi}{2k}\dfrac{(k\tilde{\eta})^{\alpha+1/2}}{(k\eta)^{\alpha-1/2}}\left[J_{\alpha-1/2}(k\tilde{\eta})Y_{\alpha-1/2}(k\eta) - J_{\alpha-1/2}(k\eta)Y_{\alpha-1/2}(k\tilde{\eta})\right]$, with $\alpha = \dfrac{3(1-w)}{2(1+3w)}$.

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…sourced by the four-point correlator of V_i

$\langle V_{\lambda_1}(\boldsymbol{q}_1,\eta_1) V_{\lambda_2}(\boldsymbol{k}_1-\boldsymbol{q}_1,\eta_1) V_{\lambda_3}(\boldsymbol{q}_2,\eta_2) V_{\lambda_4}(\boldsymbol{k}_2-\boldsymbol{q}_2,\eta_2)\rangle$, with $V_i(\boldsymbol{k},\eta) = 2 \sum_{\lambda=\pm} V_\lambda(\boldsymbol{k},\eta) e^i_\lambda(\boldsymbol{k})$.

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3

... proportional to the four-point correlator of $\Pi_i^{(V)}$

$$\langle \Pi_{i_1}^{(V)}(\boldsymbol{q}_1) \Pi_{i_2}^{(V)}(\boldsymbol{k}_1-\boldsymbol{q}_1) \Pi_{i_3}^{(V)}(\boldsymbol{q}_2) \Pi_{i_4}^{(V)}(\boldsymbol{k}_2-\boldsymbol{q}_2)\rangle,$$

which can be Wick-expanded in terms of the two-point function

$$\langle \Pi_i^{(V)}(\boldsymbol{k}) \Pi_j^{(V)}(\boldsymbol{k}')\rangle = P_{ij}(\boldsymbol{k}) \,|\, \Pi^{(V)}(k) \,|^2 \delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}').$$

$$V_i(\eta,\boldsymbol{k}) = -\frac{16\pi G \Pi_i^{(V)}(\boldsymbol{k})\eta}{a(\eta)^2 k}$$

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- Express as a convolution integral in terms of the comoving PMF power spectrum: $|\Pi^{(V)}(k)|^2 = \int d^3q P_B(k) P_B(|\boldsymbol{k} - \boldsymbol{q}|)$.

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- Express as a convolution integral in terms of the comoving PMF power spectrum: $|\Pi^{(V)}(k)|^2 = \int d^3q P_B(k) P_B(|\boldsymbol{k} - \boldsymbol{q}|)$.
- Approximate the integral analytically (with a UV-cutoff) : $|\Pi^{(V)}(k)|^2 \approx 4\pi A_B^2 \left[\frac{n_\mathrm{B} k^{2n_B+3}}{(n_B+3)(2n_B+3)} + \frac{k_\mathrm{UV}^{2n_B+3}}{2n_B+3} \right]$.

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Total parameter space: $\{H_{\rm inf}, \Delta N_{\rm reh}, s, w\} \Longrightarrow$ **Let us compute the VIGW two-point correlation function!**

The GW spectral abundance

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For freely propagating GWs on sub-horizon scales

$$\Omega_{\rm GW}(k,\eta) = \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{\rm GW}}{\mathrm{d}\ln k} \approx \frac{1}{12}\left[\frac{k}{\mathscr{H}(\eta)}\right]^2 \mathscr{P}_h(k,\eta)$$

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Present-day GW spectral abundance on scales $k \gtrsim k_{\rm reh}$:

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The GW spectral abundance on small scales can be computed at $\eta = \eta_{\rm reh}$

and related to its present-day value!

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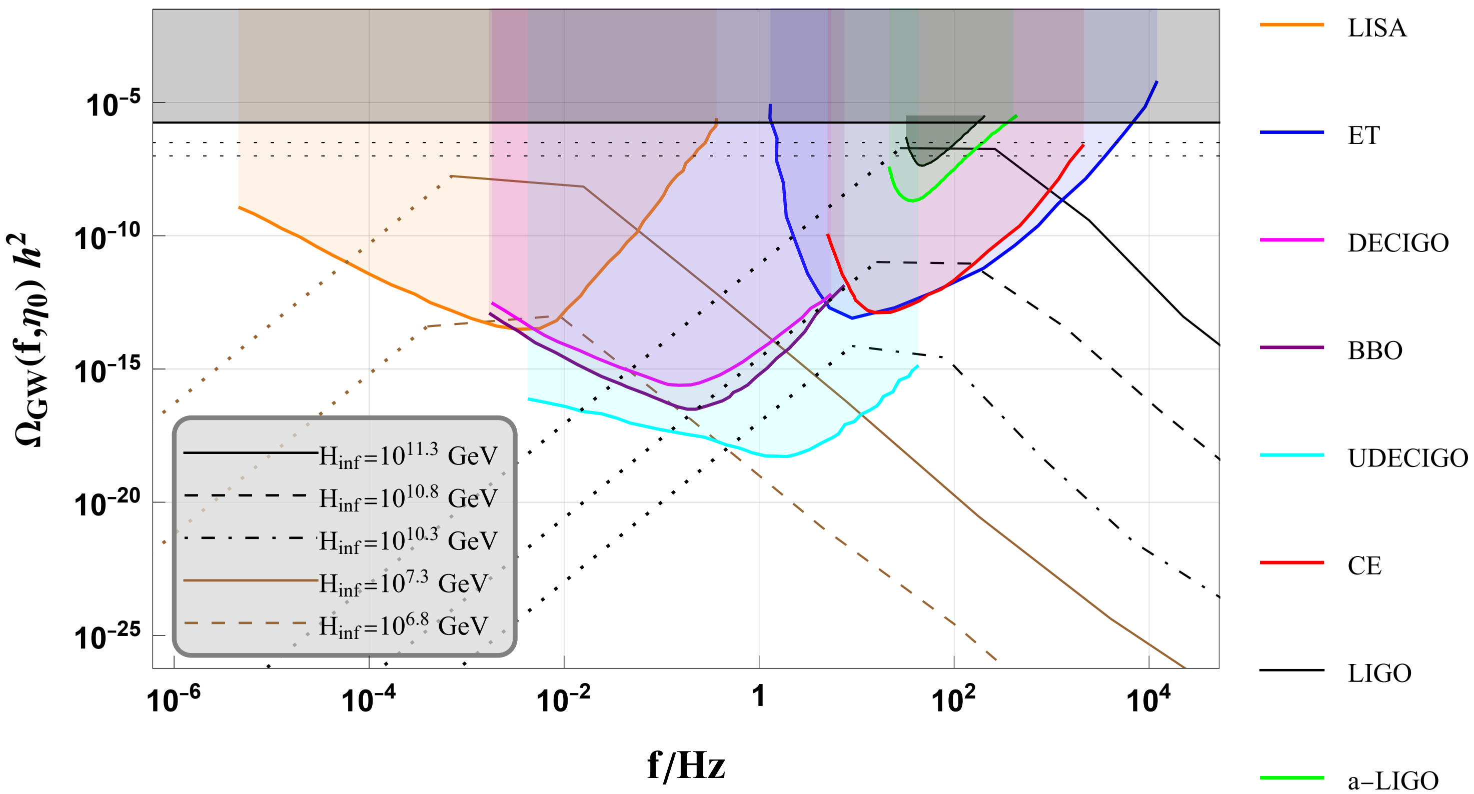
At the end, we find that a detectably large VIGW signal at small scales can be obtained for realistic parameter values!

The VIGW spectrum for kination ($s = 2$)



$\Delta N_{\rm reh} = 10$ (black) & 14 (brown)

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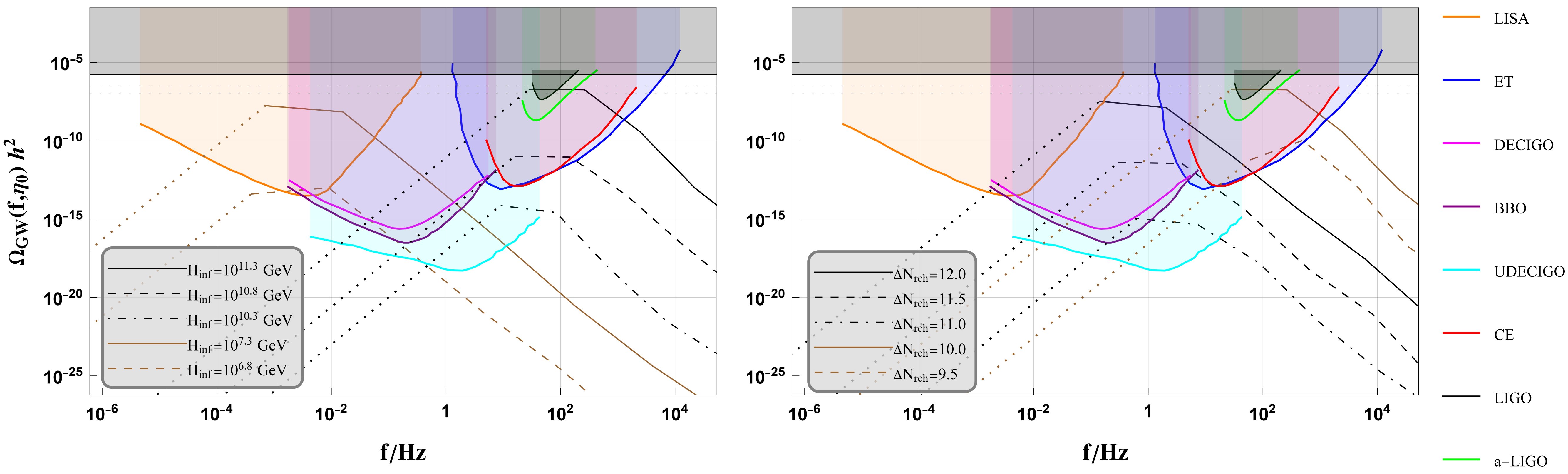

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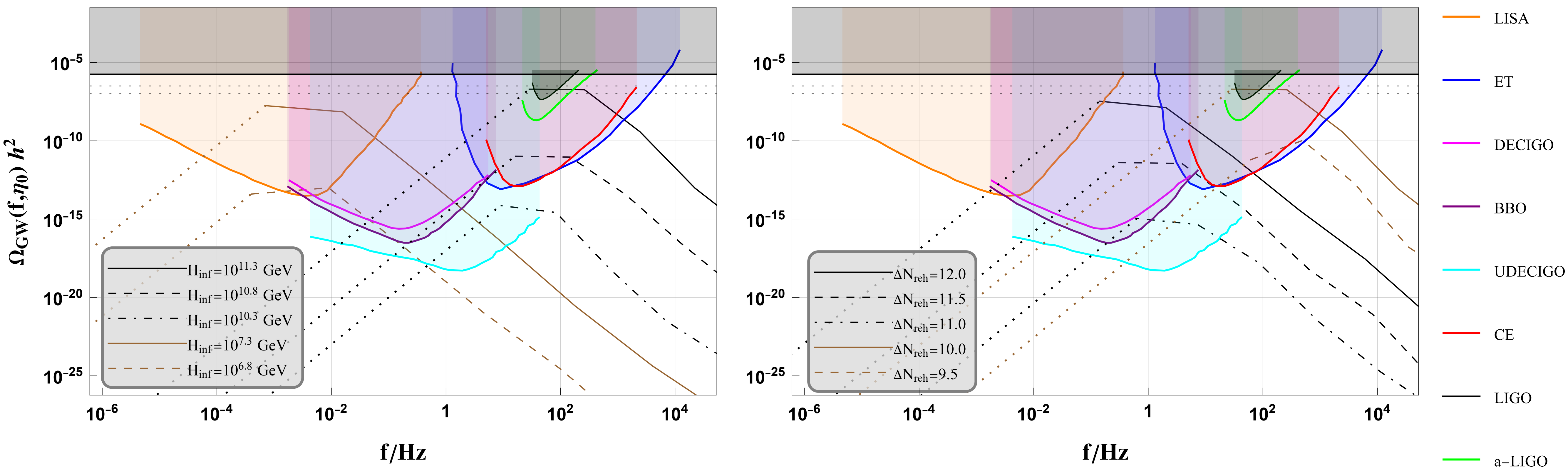

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$$f_{\rm peak} \equiv \frac{k_{\rm reh}}{2\pi} = \frac{\left(H_{\rm inf} H_{\rm eq}\right)^{1/2}}{2\pi} \left(\frac{H_0}{H_{\rm eq}}\right)^{2/3} \exp\left[-\frac{3}{4}(w_{\rm reh}+1)\Delta N_{\rm reh}\right]$$

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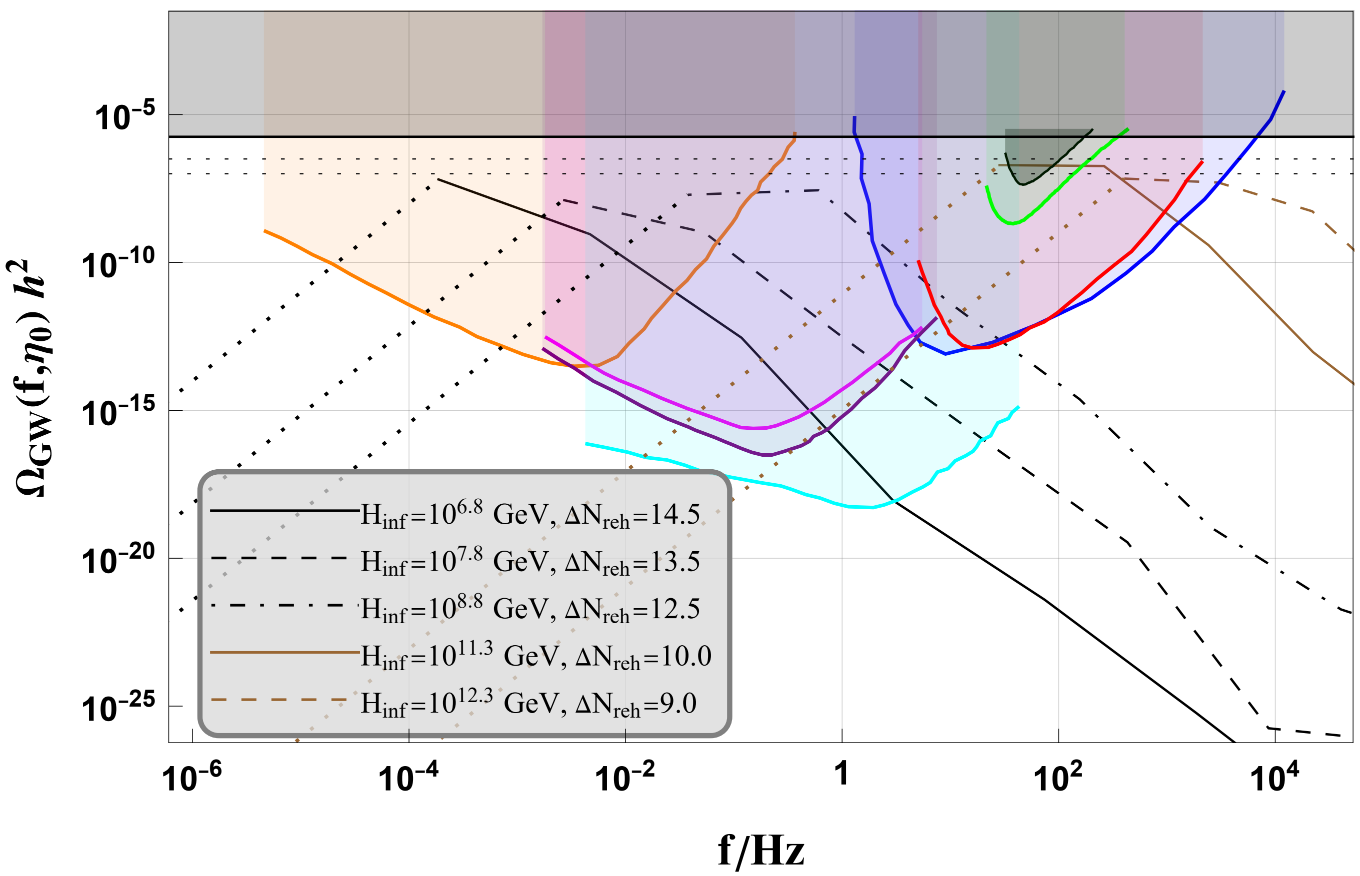

Simultaneous variation of $H_{\rm inf}$ and $\Delta N_{\rm reh}$

Constant peak GW amplitude and varying peak frequency

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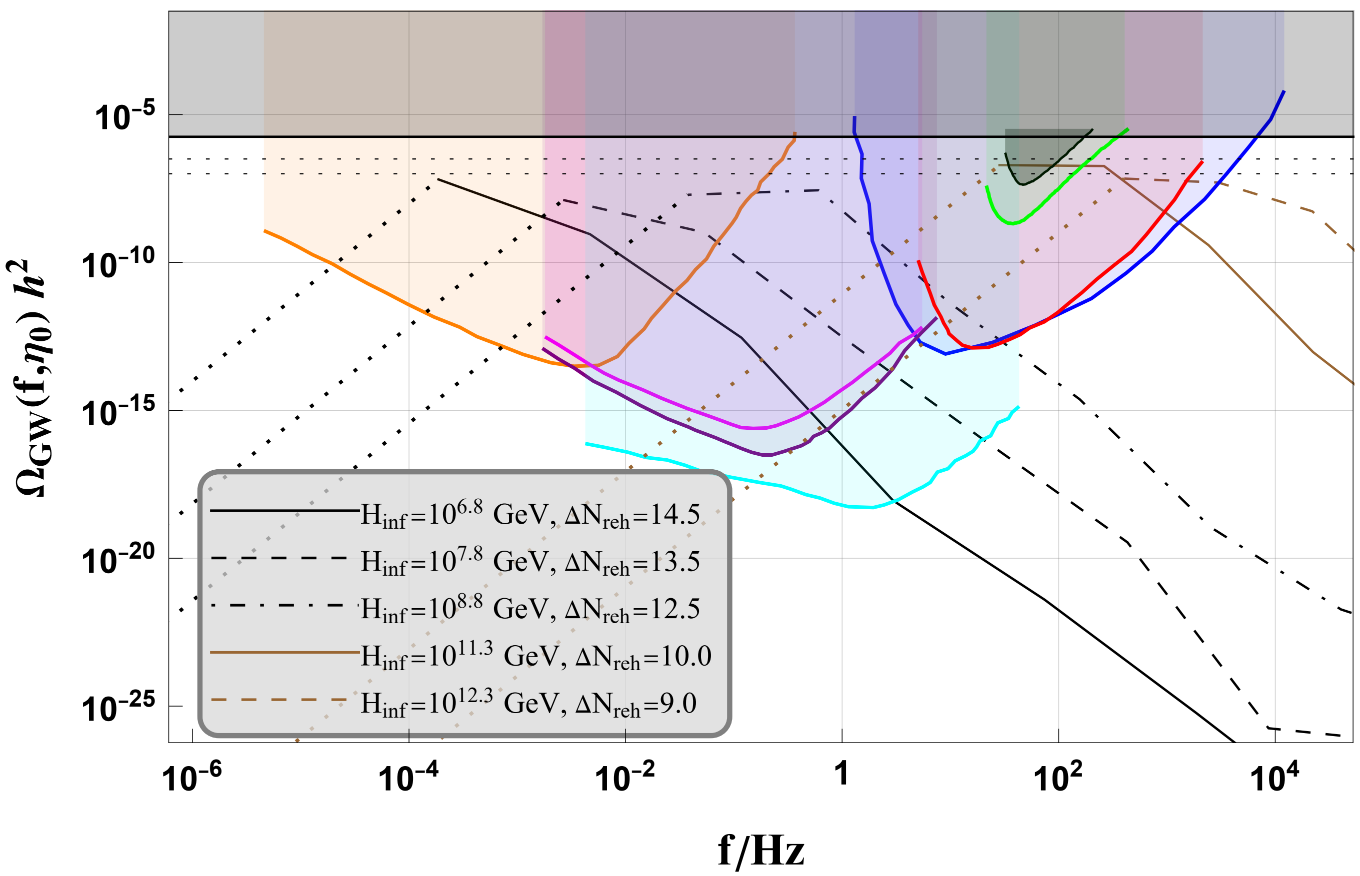

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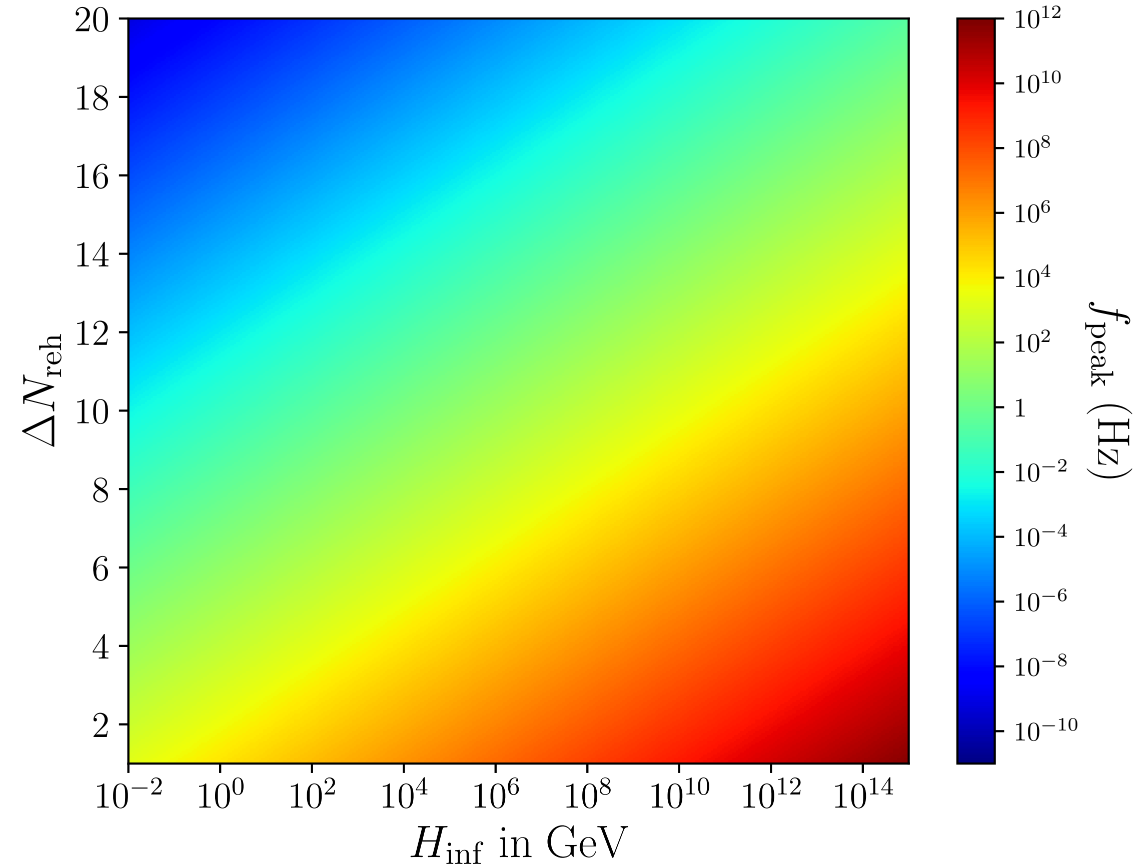

Dependence of $f_{\rm peak}$ of $H_{\rm inf}$ and $\Delta N_{\rm reh}$

(for $w = 1$)

$$f_{\rm peak} = \frac{\left(H_{\rm inf} H_{\rm eq}\right)^{1/2}}{2\pi} \left(\frac{H_0}{H_{\rm eq}}\right)^{2/3} \exp\left[-\frac{3}{2}\Delta N_{\rm reh}\right]$$

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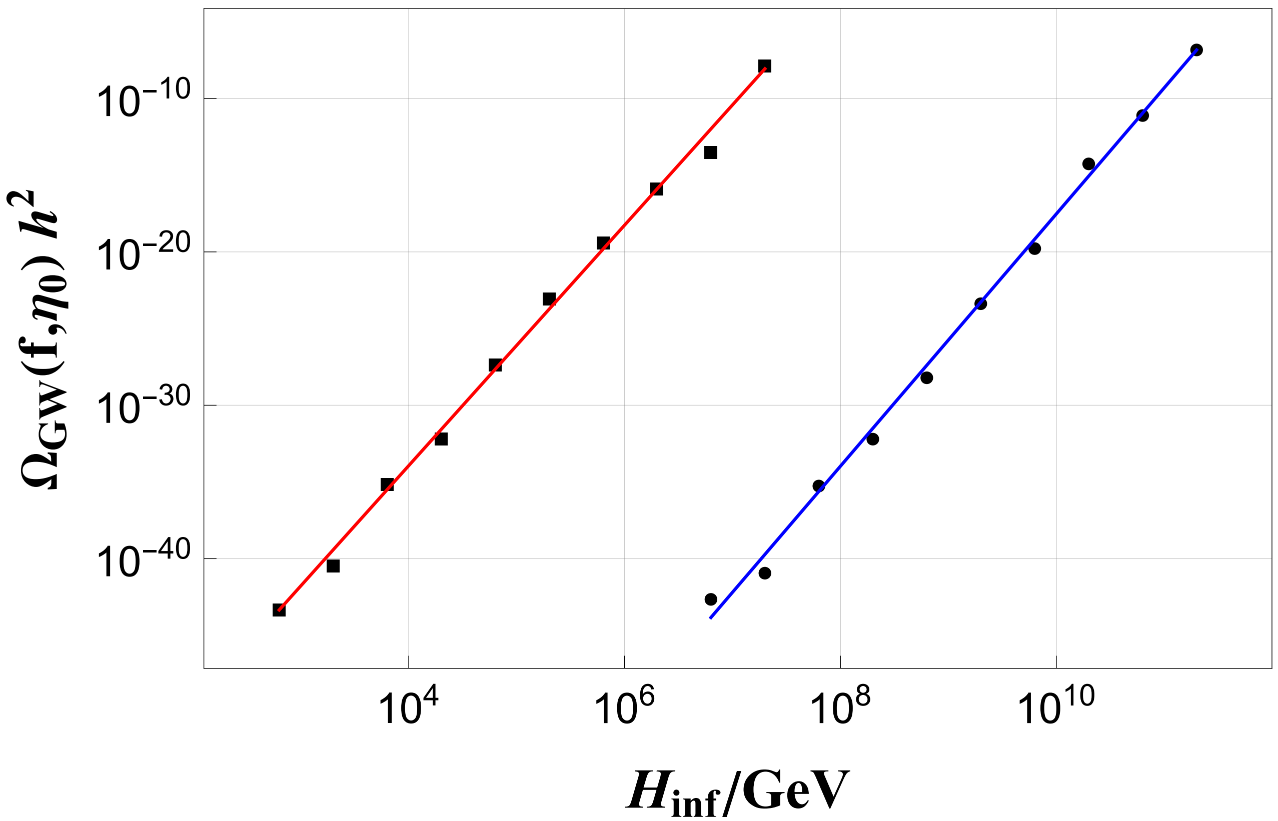

Variation of the peak GW amplitude as a function of $H_{\rm inf}$ corresponding to fixed $\Delta N_{\rm reh} = 10$ (blue) & 14 (red)

$\Longrightarrow$ **Power-law dependence**

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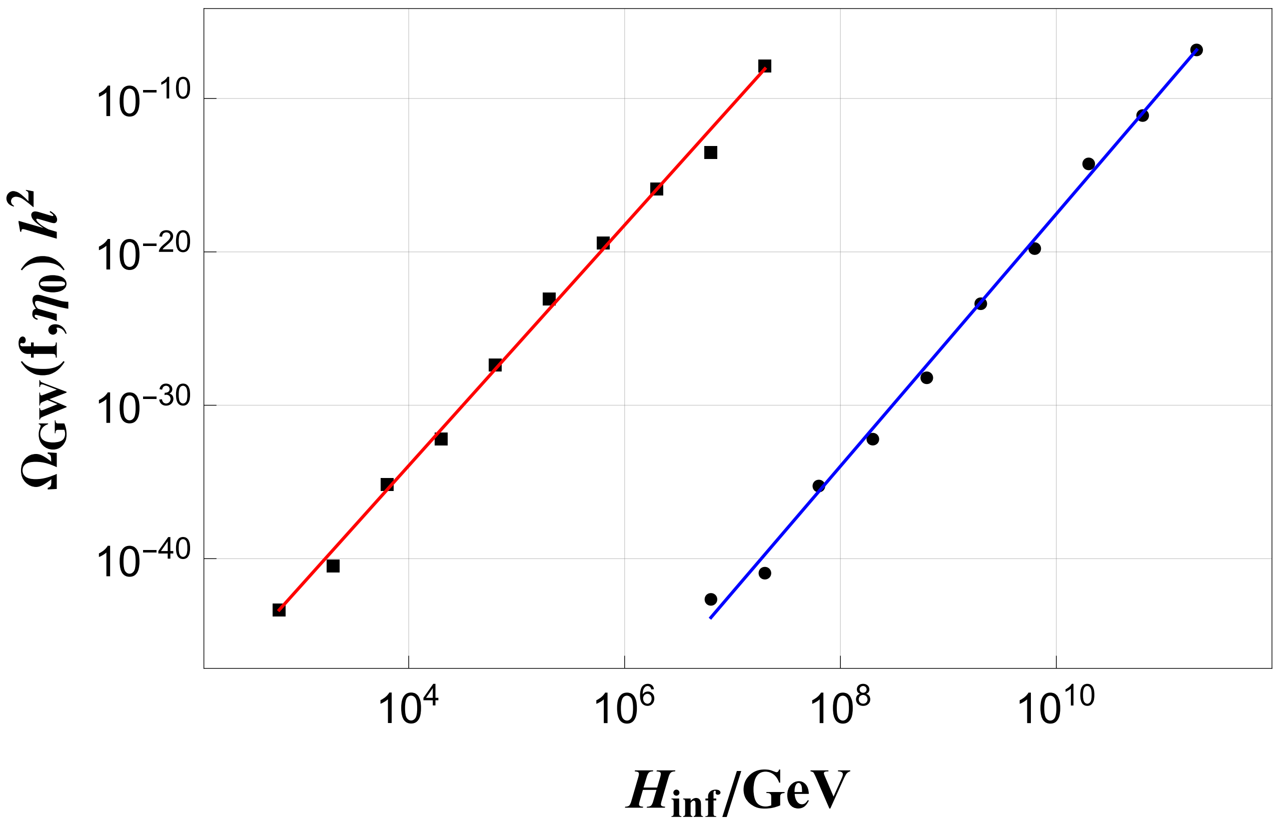

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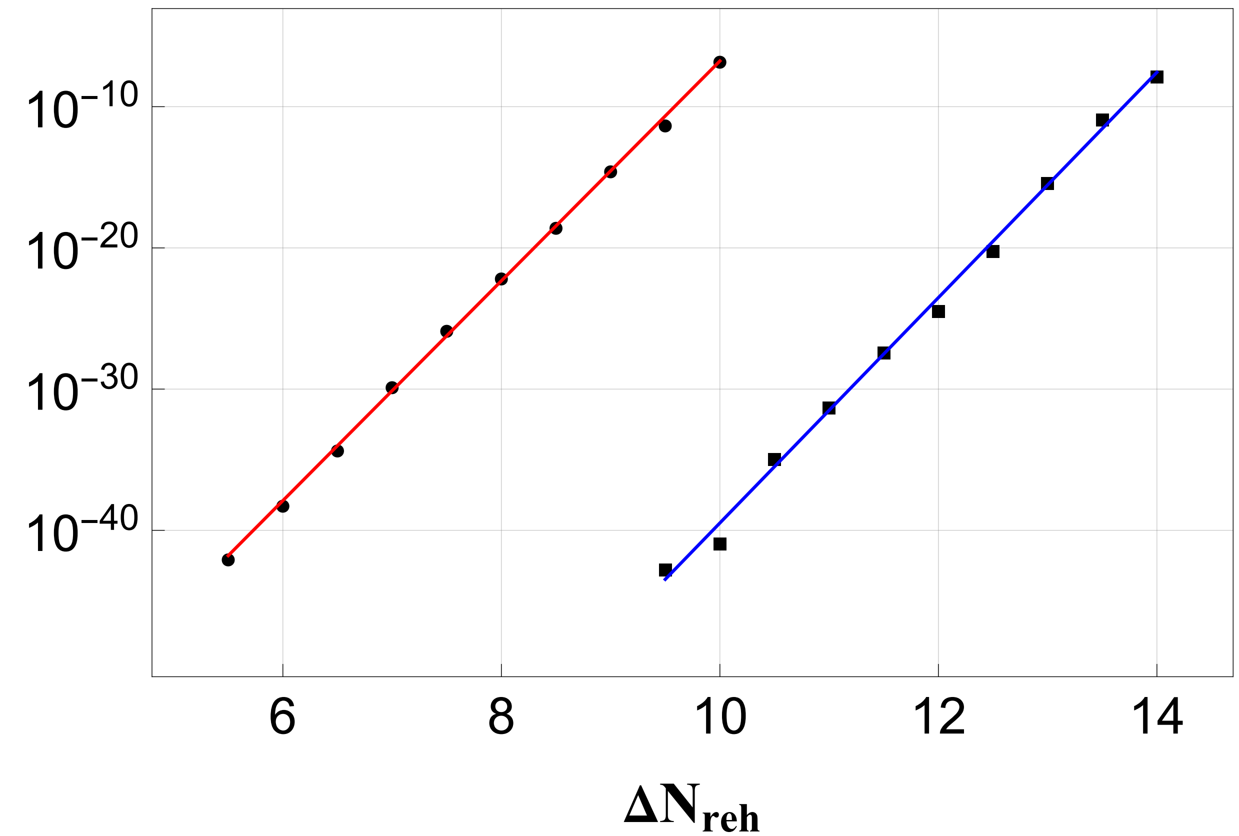

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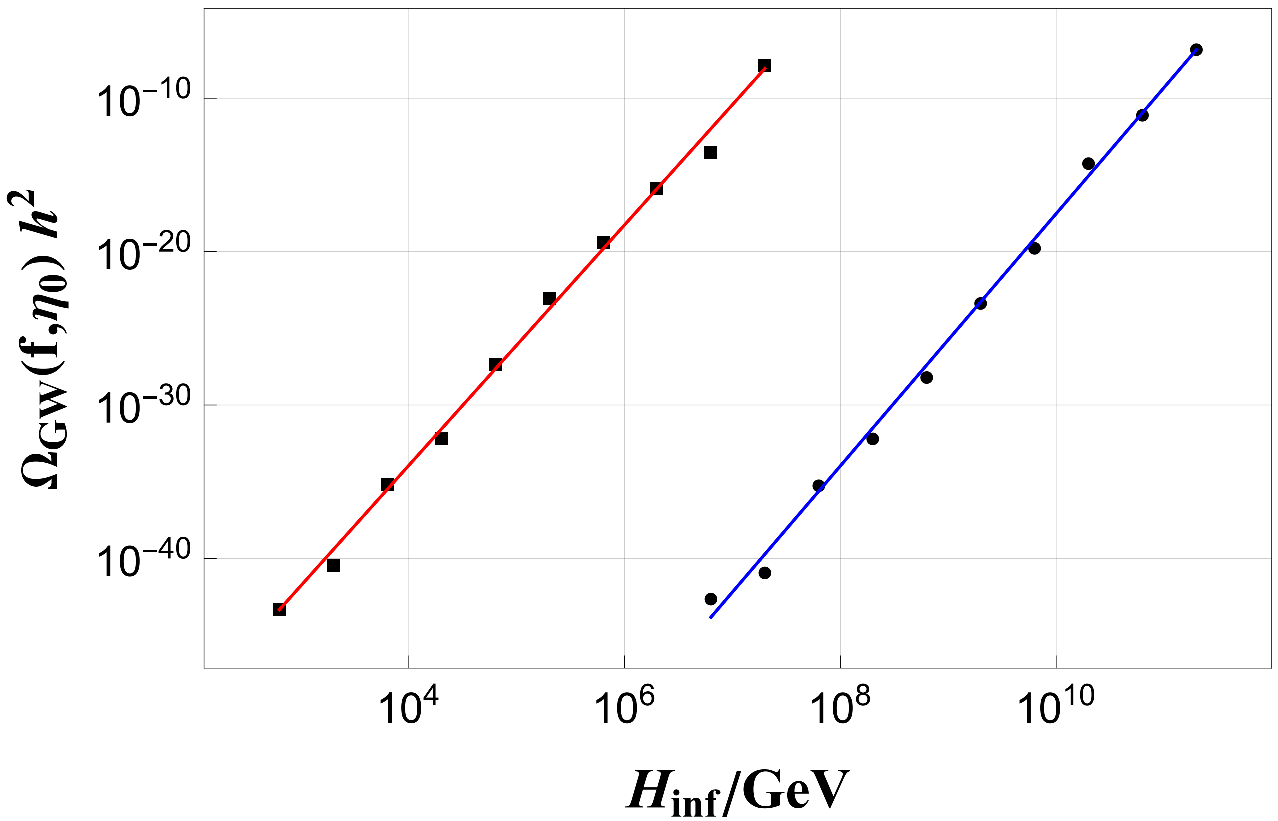

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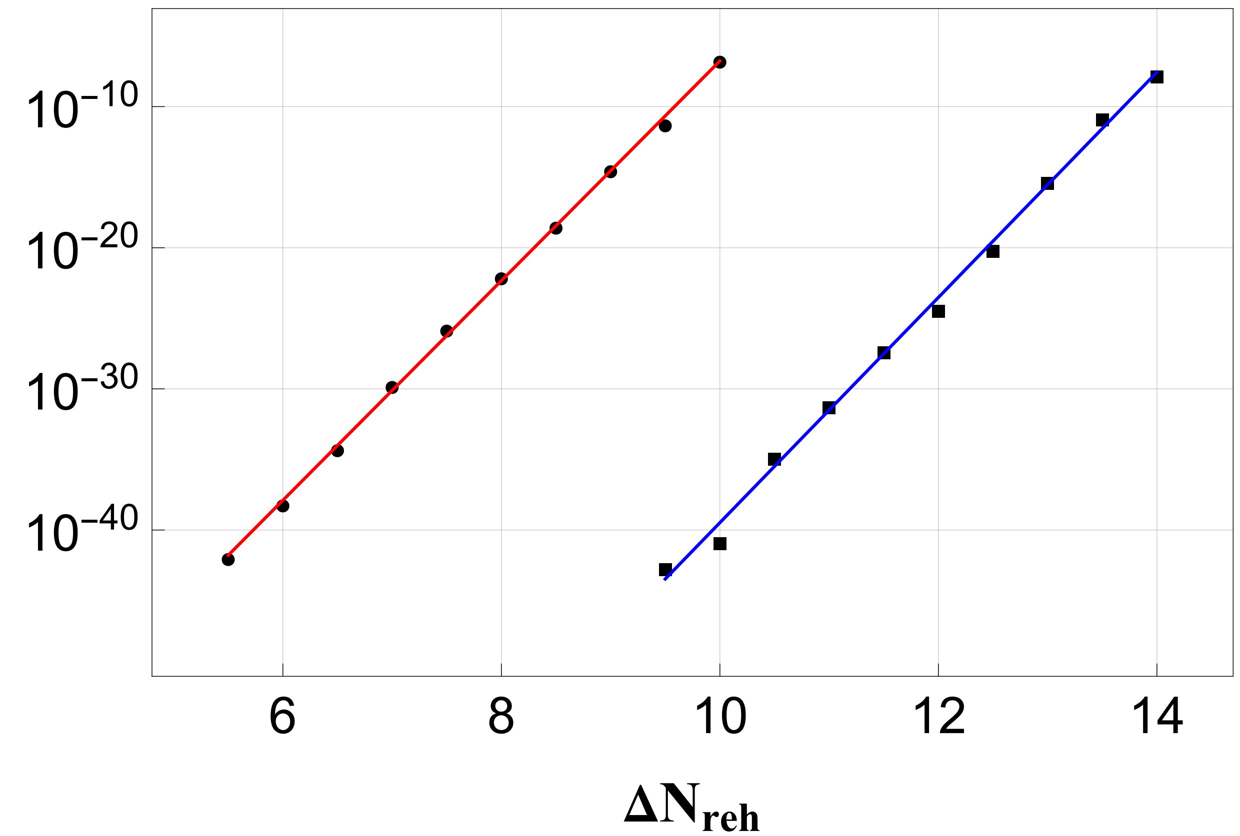

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Approximate Scaling Relation: $\Omega_{\rm GW}(f_{\rm peak}, \eta_0) \propto \left[10^{\Delta N_{\rm reh}}\left(\frac{H_{\rm inf}}{M_{\rm Pl}}\right)\right]^8$

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Future Perspectives

- Semi-analytic and/or simulation-based modelling of VIGWs during RD.
- Disentangling the VIGW spectrum from competing signals, e.g. PGWs, SIGWs.

Thanks for your attention!

Back-Up Slides

Full S+V+T source term for second-order IGWs

$$S_{ij} = - V_a\partial_a(\partial_i V_j + \partial_j V_i) + \partial_a V_i \partial_a V_j + \partial_i V_a \partial_j V_a + 2V_a \partial_i \partial_j V_a + \frac{\Delta V_i \Delta V_j}{6\mathcal{H}^2(1+w)} \quad \text{(C.1)}$$

$$- 2\partial_i\phi\partial_j\phi - 6\partial_i\psi\partial_j\psi - 4\left(\phi\partial_i\partial_j\phi + \psi\partial_i\partial_j\psi\right) + 2\left(\partial_i\phi\partial_j\psi + \partial_j\phi\partial_i\psi\right)$$

$$+ \frac{8}{3\mathcal{H}^2(1+w)}\left[\mathcal{H}^2\partial_i\phi\partial_j\phi + \mathcal{H}\left(\partial_i\phi\partial_j\psi' + \partial_j\phi\partial_i\psi'\right) + \partial_i\psi'\partial_j\psi'\right] \quad \text{(C.2)}$$

$$+ 4h'_{ia}h'_{ja} - 2\partial_i h_{ab}\partial_j h_{ab} + 4h_{ab}\partial_b\left(\partial_i h_{aj} + \partial_j h_{ai}\right)$$

$$+ 4\partial_b h_{ia}\left(\partial_a h_{bj} - \partial_b h_{aj}\right) - 4h_{ab}\left(\partial_i\partial_j h_{ab} + \partial_a\partial_b h_{ij}\right) + 12\mathcal{H}^2 w^{(1)} h_{ij} \quad \text{(C.3)}$$

$$- \left(\phi' + \psi' + 4\mathcal{H}\phi\right)\left(\partial_i V_j + \partial_j V_i\right) - 2\phi\left(\partial_i V'_j + \partial_j V'_i\right) + 4\mathcal{H}\left(V_i\partial_j\psi + V_j\partial_i\psi\right)$$

$$+ 2\left[(V_i\partial_j\psi)' + (V_j\partial_i\psi)'\right] - \frac{2}{3\mathcal{H}(1+w)}\left(\Delta V_i\partial_j + \Delta V_j\partial_i\right)\left(\phi + \frac{\psi'}{\mathcal{H}}\right) \quad \text{(C.4)}$$

$$+ 4\phi\left(h''_{ij} + 2\mathcal{H}h'_{ij} + 6w\mathcal{H}^2 h_{ij}\right) - 2\left(\phi' - \psi'\right)h'_{ij} - 8\mathcal{H}\left(\phi' + 3\psi'\right)h_{ij}$$

$$- 12\psi'' h_{ij} + 4\left(\psi\Delta h_{ij} - \Delta\phi h_{ij} + 2\Delta\psi h_{ij}\right) - 4\left(h_{ia}\partial_a\partial_j\psi + h_{ja}\partial_a\partial_i\psi\right)$$

$$+ 2\partial_a h_{ij}\partial_a\left(\phi + 3\psi\right) - 2\left(\partial_i h_{aj} + \partial_j h_{ia}\right)\partial_a\left(\phi + \psi\right) \quad \text{(C.5)}$$

$$+ 4V_a\partial_a h'_{ij} + 2\left(V'_a + 4\mathcal{H}V_a\right)\partial_a h_{ij} - 2\left(h'_{ia}\partial_a V_j + h'_{aj}\partial_a V_i\right)$$

$$- 2\left[(V_a\partial_i h_{aj})' + (V_a\partial_j h_{ia})'\right] - 4\mathcal{H}V_a\left(\partial_i h_{aj} + \partial_j h_{ia}\right)\ , \quad \text{(C.6)}$$

Our focus is on (C.1); pure SIGWs are sourced by (C.2);

pure tensor-induced tensors result from (C.3); rest are SVT cross-terms.

Analytic expressions for $\langle S_+ S_+ \rangle$

$$
\begin{aligned}
&\langle S_+(\boldsymbol{k}_1,\eta_1)S_+(\boldsymbol{k}_2,\eta_2)\rangle_A \\
&= \delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\times\frac{8(16\pi G)^4\eta_1^2\eta_2^2}{a(\eta_1)^4 a(\eta_2)^4}\sum_{\lambda_1,\lambda_2=\pm}\int\frac{d^3q_1}{(2\pi)^{3/2}}\frac{|\Pi^{(V)}(q_1)|^2|\Pi^{(V)}(|\boldsymbol{k}_1-\boldsymbol{q}_1|)|^2}{q_1^2|\boldsymbol{k}_1-\boldsymbol{q}_1|^2} \\
&\times\Bigg[\frac{1}{16}\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_1^2}\right)\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_2^2}\right) \\
&\times(1+\lambda_1\cos\theta)^2\left(1+\lambda_2\frac{k_1-q_1\cos\theta}{|\boldsymbol{k}_1-\boldsymbol{q}_1|}\right)^2+\frac{1}{16}q_1^4\sin^4\theta\left(1-\lambda_1\lambda_2\frac{k_1-q_1\cos\theta}{|\boldsymbol{k}_1-\boldsymbol{q}_1|}\right)^2 \\
&+\frac{1}{4}k_1^2q_1^2\left(1+\lambda_2\frac{k_1-q_1\cos\theta}{|\boldsymbol{k}_1-\boldsymbol{q}_1}\right)^2-2\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_1^2}\right) \\
&\times\Re\left[\left(q_{1a}e_-^a(\boldsymbol{k}_1)^*\right)^2e_-^\ell(\boldsymbol{k}_1)e_{\lambda_1}^\ell(\boldsymbol{q}_1)e_-^m(\boldsymbol{k}_1)e_{\lambda_2}(\boldsymbol{k}_1-\boldsymbol{q}_1)e_{\lambda_1}^d(\boldsymbol{q})^*e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right] \\
&+\frac{1}{2}\left(1+\lambda_2\frac{k_1-q_1\cos\theta}{|\boldsymbol{k}_1-\boldsymbol{q}_1|}\right)^2\times\Re\left[k_{1c}e_{\lambda_1}^c(\boldsymbol{q}_1)q_{1\ell}e_-^\ell(\boldsymbol{k}_1)e_-^a(\boldsymbol{k}_1)^*e_{\lambda_1}^a(\boldsymbol{q}_1)^*\right] \\
&-2q_1^2\sin^2\theta\times\Re\left[k_{1c}e_{\lambda_1}^c(\boldsymbol{q}_1)q_{1b}e_-^b(\boldsymbol{k}_1)^*e_-^m(\boldsymbol{k}_1)e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)e_{\lambda_1}^d(\boldsymbol{q}_1)^*e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right]\Bigg]
\end{aligned}
$$

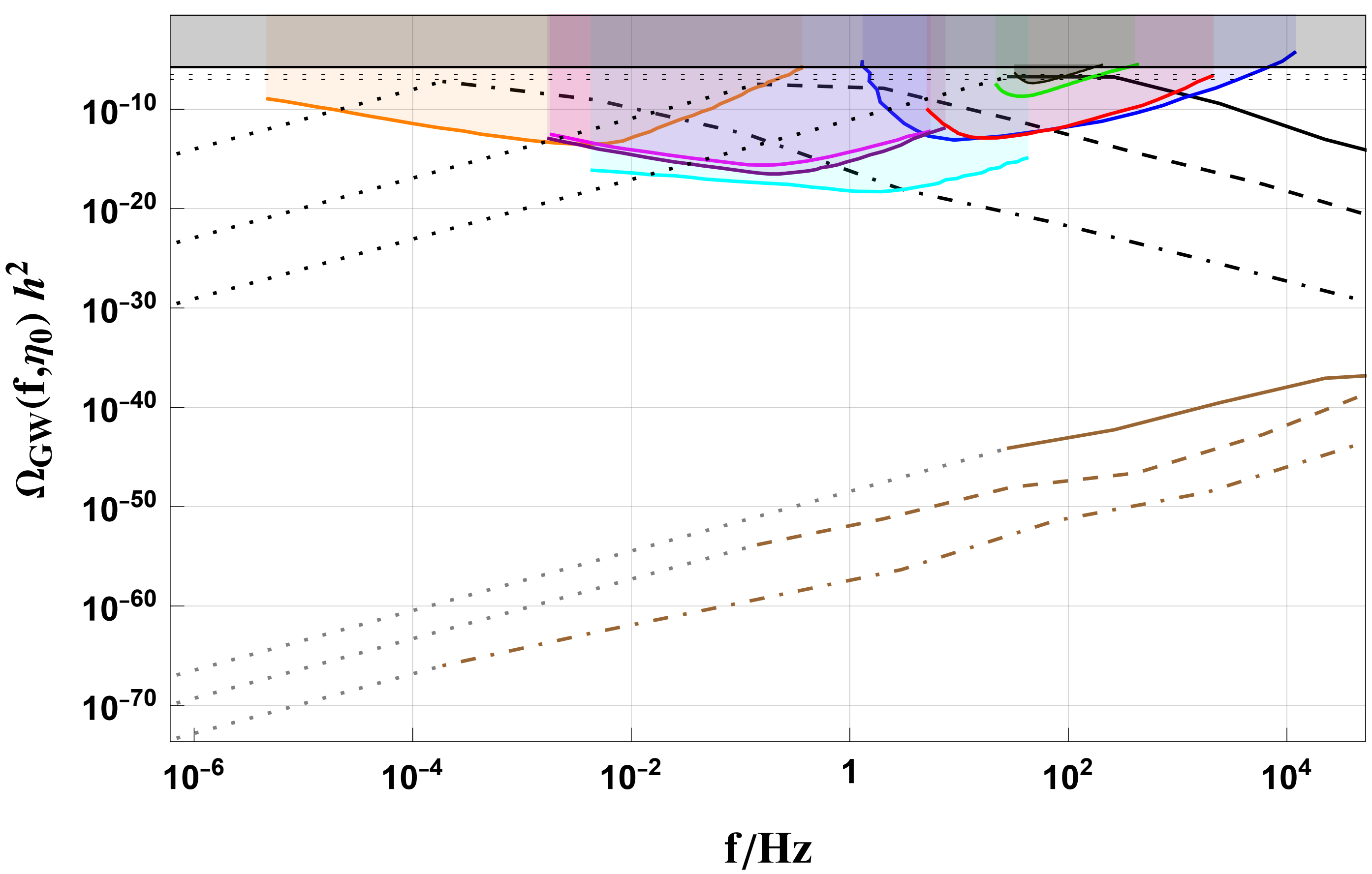
$$
\begin{aligned}
&\langle S_+(\boldsymbol{k}_1,\eta_1)S_+(\boldsymbol{k}_2,\eta_2)\rangle_B \\
&= \delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\times\frac{8(16\pi G)^4\eta_1^2\eta_2^2}{a(\eta_1)^4 a(\eta_2)^4}\sum_{\lambda_1,\lambda_2=\pm}\int\frac{d^3q_1}{(2\pi)^{3/2}}\frac{|\Pi^{(V)}(q_1)|^2|\Pi^{(V)}(|\boldsymbol{k}_1-\boldsymbol{q}_1|)|^2}{q_1^2|\boldsymbol{k}_1-\boldsymbol{q}_1|^2} \\
&\times\Bigg[\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_1^2}\right)\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_2^2}\right) \\
&\times|e_{\lambda_1}^\ell(\boldsymbol{q}_1)e_-^\ell(\boldsymbol{k}_1)|^2|e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)e_-^m(\boldsymbol{k}_1)|^2+|q_{1\ell}e_-^\ell(\boldsymbol{k}_1)|^4|e_{\lambda_1}^c(\boldsymbol{q}_1)e_{\lambda_2}^c(\boldsymbol{k}_1-\boldsymbol{q}_1)|^2 \\
&-4|q_{1\ell}e_-^\ell(\boldsymbol{k}_1)|^2k_{1c}e_{\lambda_1}^c(\boldsymbol{q}_1)k_{1d}e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*e_-^m(\boldsymbol{k}_1)e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)e_-^b(\boldsymbol{k}_1)^*e_{\lambda_1}^b(\boldsymbol{q}_1)^* \\
&-2\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_1^2}\right) \\
&\times\Re\left[\left(q_{1a}e_-^a(\boldsymbol{k}_1)^*\right)^2e_-^\ell(\boldsymbol{k}_1)e_{\lambda_1}^\ell(\boldsymbol{q}_1)e_-^m(\boldsymbol{k}_1)e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)e_{\lambda_1}^d(\boldsymbol{q}_1)^*e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right] \\
&+2\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_1^2}\right) \\
&\times|e_-^b(\boldsymbol{k}_1)e_{\lambda_1}^b(\boldsymbol{q}_1)|^2\left(k_{1d}e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right)\left(q_{1a}e_-^a(\boldsymbol{k}_1)^*\right)\left(e_-^m(\boldsymbol{k}_1)e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)\right) \\
&-2\left(\boldsymbol{q}_1.(\boldsymbol{q}_1-\boldsymbol{k}_1)+\frac{q_1^2|\boldsymbol{q}_1-\boldsymbol{k}_1|^2}{6(1+w)\mathcal{H}_2^2}\right) \\
&\times|e_-^a(\boldsymbol{k}_1)e_{\lambda_2}^a(\boldsymbol{k}_1-\boldsymbol{q}_1)|^2\left(k_{1c}e_{\lambda_1}^c(\boldsymbol{q}_1)\right)\left(q_{1\ell}e_-^\ell(\boldsymbol{k}_1)\right)\left(e_-^b(\boldsymbol{k}_1)^*e_{\lambda_1}^b(\boldsymbol{q}_1)^*\right) \\
&-2\left(q_{1\ell}e_-^\ell(\boldsymbol{k}_1)\right)^2\left(k_{1d}e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right)\left(q_{1a}e_-^a(\boldsymbol{k}_1)^*\right)\left(e_-^b(\boldsymbol{k}_1)^*e_{\lambda_1}^b(\boldsymbol{q}_1)^*\right)\left(e_{\lambda_1}^c(\boldsymbol{q}_1)e_{\lambda_2}^c(\boldsymbol{k}_1-\boldsymbol{q}_1)\right) \\
&+2\left(q_{1a}e_-^a(\boldsymbol{k}_1)^*\right)^2\left(k_{1c}e_{\lambda_1}^c(\boldsymbol{q}_1)\right)\left(q_{1\ell}e_-^\ell(\boldsymbol{k}_1)\right)\left(e_-^m(\boldsymbol{k}_1)e_{\lambda_2}^m(\boldsymbol{k}_1-\boldsymbol{q}_1)\right)\left(e_{\lambda_1}^d(\boldsymbol{q}_1)^*e_{\lambda_2}^d(\boldsymbol{k}_1-\boldsymbol{q}_1)^*\right)\Bigg]
\end{aligned}
$$

$$
\langle h_{ij}h^{ij}\rangle \sim \langle S_+S_+\rangle+\langle S_-S_-\rangle+\langle S_+S_-\rangle+\langle S_-S_+\rangle = 2\left(\langle S_+S_+\rangle+\langle S_+S_-\rangle\right)
$$

VIGWs vs first order PMF sourced GWs

Apart from vector perturbations, PMFs can also directly source tensor perturbations.

$$\mathscr{P}_T^{(B)}(k,\eta) = 64G^2k^3\,|\,\Pi^{(T)}(k)\,|^2 \left[\int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{a(\tilde{\eta}_1)^2} g_k(\eta,\tilde{\eta})\right]^2$$



- Comparison of VIGW spectrum vs first-order PMF-sourced GW spectrum produced during a post-inflationary kination epoch for identical combinations of parameters:

- $H_{\rm inf} = 10^{6.8}$GeV & $\Delta N_{\rm reh} = 14.5$ (dot-dashed);
  $H_{\rm inf} = 10^{9.3}$GeV & $\Delta N_{\rm reh} = 12.0$ (dashed);
  $H_{\rm inf} = 10^{11.3}$GeV & $\Delta N_{\rm reh} = 10.0$ (solid).

- **Can be explained based on the following observations:**
  - PMFs decay whereas vector modes do not.
  - VIGW spectrum receives η and k-integrated contributions.

What about $s \neq 2$?

- For example, choosing $s = 1 \implies$ merely one order reduction in peak amplitude.
- This is because for $2n_B + 3 > 0$, the UV-cutoff term dominates the analytic expression for $|\Pi^{(V)}(k)|^2$, whose index therefore becomes independent of the magnetic spectral index.
- Negative and non-integer values of s are less favoured from an inflationary magnetogenesis perspective.

What about $w \neq 1$?

- Vector modes start decaying for $w < 1 \implies$ VIGW production is less efficient.
- The reduction is quite significant due to four-point dependence on vectors + time integrated effect.
- Larger numerical instabilities due to fast-oscillating Bessel functions $\implies$ something to revisit in future!

Avoiding magnetic backreaction

Backreaction constraint at the end of kination epoch

$$\int_{k_{\rm reh}}^{k_{\rm inf}} \frac{d\rho_B(k,\eta_{\rm reh})}{d\ln k} d\ln k \; < \; \frac{\pi^2}{30} g_{\rm reh} T_{\rm reh}^4$$

- For $s = 2 : \left(H_{\rm inf}/{\rm GeV}\right) e^{\Delta N_{\rm reh}} < 10^{51} \implies$ satisfied by all of our choices
- For $s = 4 : \left(H_{\rm inf}/{\rm GeV}\right) e^{3\Delta N_{\rm reh}} < 10^{50}$
- For arbitrary $s : \dfrac{2^{2(s-1)}}{\pi(3-s)} \Gamma\left(s + \dfrac{1}{2}\right)^2 a_{\rm inf}^4 \left(H_{\rm inf}/\text{GeV}\right)^4 \left[1 - e^{-4(3-s)\Delta N_{\rm reh}}\right] \; < \; 10^{-42}$

For larger values of s, the backreaction constraint may rule out parts of the $\{H_{\rm inf}, \Delta N_{\rm reh}\}$ parameter space.