EXPLOITING THE EFFECTIVE-ONE-BODY APPROACH FOR LARGE-MASS-RATIO BLACK HOLE BINARIES

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OUTLINE

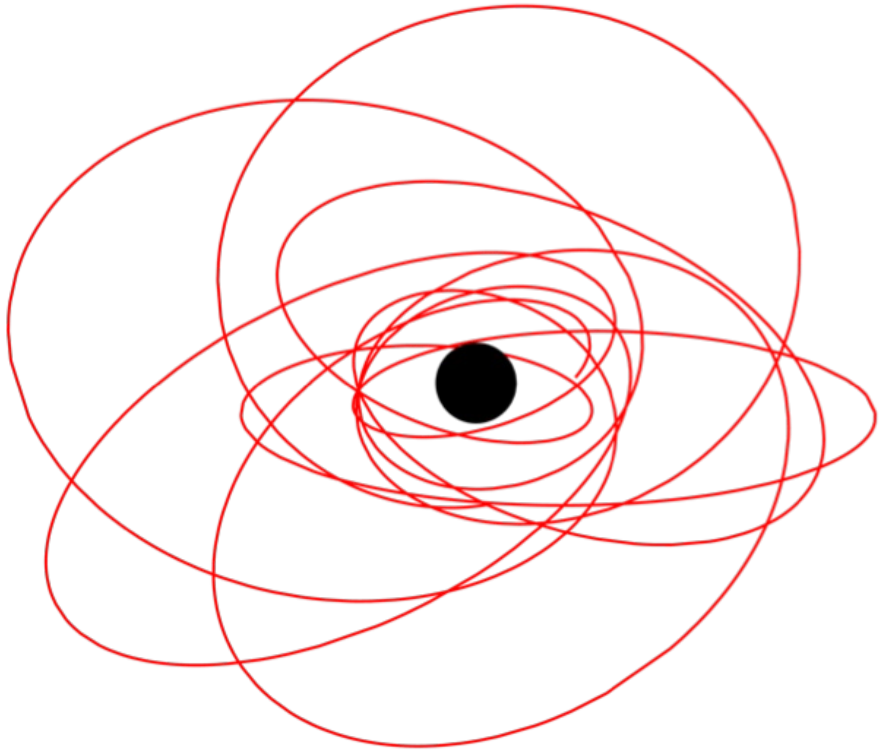
- Motivation
- Intro to the effective-one-body (EOB) approach and the model **TEOBResumS**
- v1: large-mass-ratio version for **inspirals** only
  benchmarked to gravitational self-force (GSF) results
- v2: large-mass-ratio version tuned to numerical relativity (NR)
  complete inspiral-merger-ringdown model
- Open-source code for waveforms with eccentricity and precession
- Conclusions & future work

LARGE-MASS-RATIO INSPIRALS

large-mass-ratio (LMR) inspirals
(intermediate + extreme)

$$q \equiv \frac{m_1}{m_2} \sim 10^2 - 10^6$$

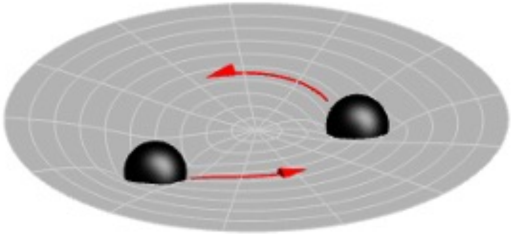
- sources for the next generation of gravitational-wave detectors
- regime not fully covered by any approach to the two-body problem
- need the **synergy** of different formalisms



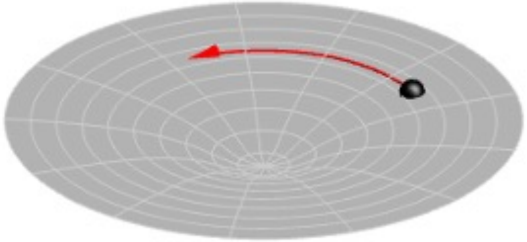
Created with the Black Hole Perturbation Toolkit

THE EFFECTIVE-ONE-BODY FORMALISM

A. Buonanno, T. Damour 1998



post-Newtonian (PN)
Hamiltonian
for the two-body problem



effective Hamiltonian
(motion in a deformed
Schwarzschild/Kerr metric)

$$\frac{H_{\rm EOB}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

$\mu = m_1 m_2/M, \quad \nu = \mu/M$

TEOBRESUMS

EOB waveform model:

- Resumming PN results
- Two branches:
  GIOTTO – quasi-circular
  DALÍ – eccentric
- Aligned/precessing spins
- Comparable masses: informed & benchmarked with NR
- For large mass ratios: GSF takes the role of NR

HAMILTONIAN

$$ds^2_{\text{eff}} = g^{\text{eff}}_{\mu\nu} dx^\mu_{\text{eff}} dx^\nu_{\text{eff}} = -A(r)dt^2 + B(r)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$g^{\mu\nu}_{\text{eff}} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mu^2 c^2 + Q = 0$$

- Effective Hamiltonian (G = c = 1):

$$\hat{H}_{\text{eff}} = \underbrace{\sqrt{p^2_{r_*} + A\left(1 + \frac{p^2_\varphi}{r^2_c} + Q\right)}}_{\text{orbital}} + \underbrace{p_\varphi\left(G_S \hat{S} + G_{S_*}\hat{S}_*\right)}_{\text{spin-orbit}}$$

A, $D \equiv A\,B$ and Q are the three EOB potentials

$$r = \frac{R}{M}, \quad p_{r_*} = \frac{P_{R_*}}{\mu}, \quad p_\varphi = \frac{P_\varphi}{\mu M}, \quad t = \frac{T}{M}$$

dimentionless coordinates and momenta

$$p_{r_*} = (A/B)^{1/2} p_r$$

tortoise radial momentum

DYNAMICS & WAVEFORM

G = c = 1

- The equations of motion are complemented by the radiation reaction:

$$
\begin{cases}
\dfrac{d\varphi}{dt} = \dfrac{\partial \hat{H}_{\text{EOB}}}{\partial p_\varphi} = \Omega \\
\dfrac{dr}{dt} = \left(\dfrac{A}{B}\right)^{1/2} \dfrac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}} \\
\dfrac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi \\
\dfrac{dp_{r_*}}{dt} = -\left(\dfrac{A}{B}\right)^{1/2} \dfrac{\partial \hat{H}_{\text{EOB}}}{\partial r} + \hat{\mathcal{F}}_r
\end{cases}
$$

- Waveform: $h_+ - ih_\times = \dfrac{1}{\mathcal{D}_L} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}$

we consider h_{22}

RADIATION REACTION

$$\hat{\mathcal{F}}_r = \text{correction} \cdot \hat{\mathcal{F}}_\varphi$$

$$\hat{\mathcal{F}}_\varphi = \hat{\mathcal{F}}_\varphi^\infty + \hat{\mathcal{F}}_\varphi^{\mathrm{H}}$$

asymptotic contribution   horizon contribution

$$\hat{\mathcal{F}}_\varphi^{\infty,\mathrm{H}} = \text{prefactor} \cdot \sum_{\ell m} \mathcal{F}_{\ell m}^{\infty,\mathrm{H}}$$

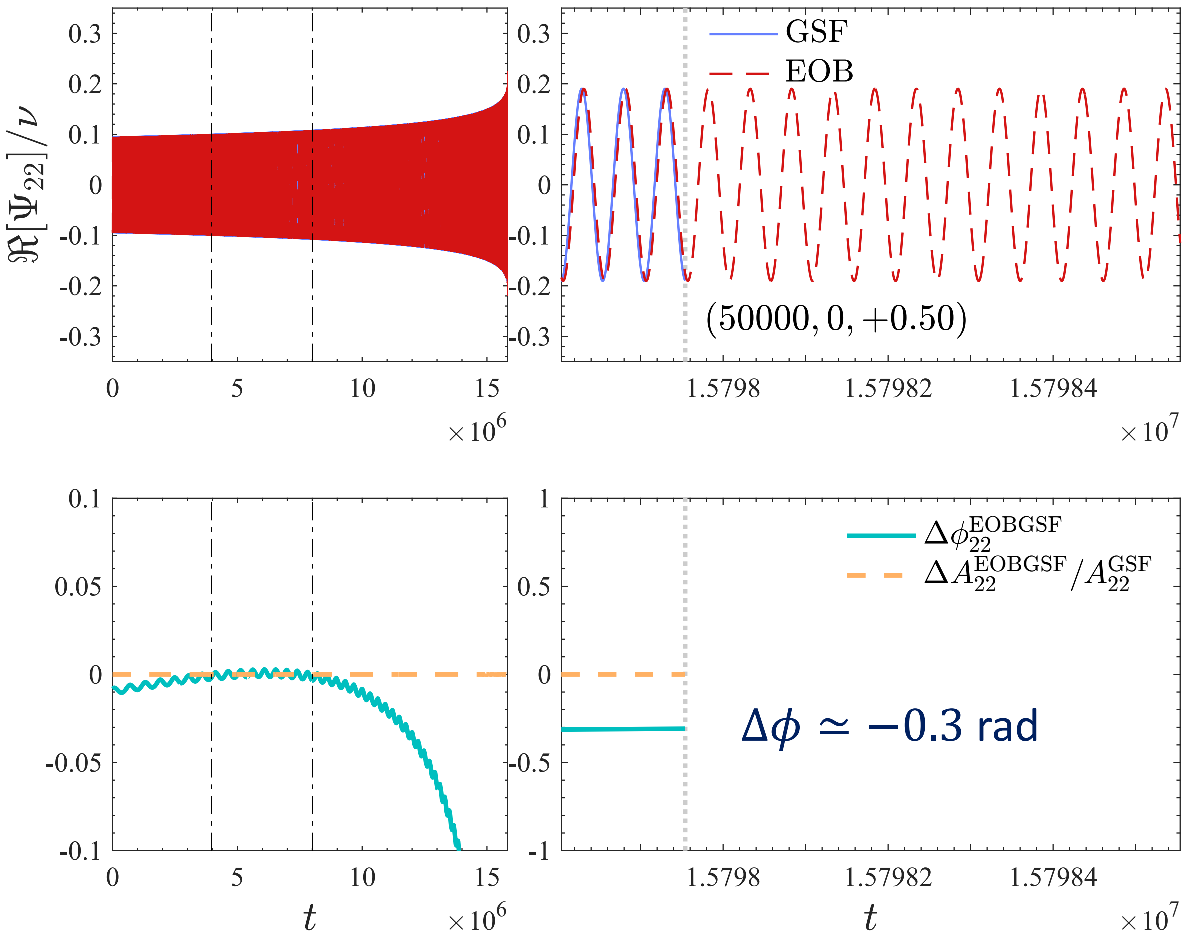
$$\mathcal{F}_{\ell m}^{\infty,\mathrm{H}} \propto \left| \hat{h}_{\ell m}^{\infty,\mathrm{H}} \right|^2$$

evaluated at every t

$$\hat{h}_{\ell m}^{\infty,\mathrm{H}} = \hat{S}_{\mathrm{eff}}^{\epsilon} \, \hat{h}_{\ell m}^{\mathrm{tail}} \left(\rho_{\ell m}^{\infty,\mathrm{H}} \right)^{\ell}$$

residual amplitude corrections

FIRST VERSION OF THE MODEL



- Dissipative sector:
  $\rho_{\ell m}^{\infty}$: 22PN + ν-info up to 3PN
  $\rho_{\ell m}^{\rm H}$: hybrid (PN/fit)
- Conservative sector: linear-in-ν, GSF-tuned potentials with light-ring singularity
  ➡ no ringdown!
- Tailored spin-orbit sector

arXiv:2208.02055, arXiv:2310.13578, arXiv:2406.04108

NEW RESUMMATION OF THE POTENTIALS

- Old way of resumming:

NR-tuned

$$A_{\rm orb}^{\rm 5PN}(u) = 1 - 2u + 2\nu u^3 + a_4 \nu u^4 + \nu \left[a_5^c(\nu) + a_5^{\log} \log u\right] u^5 + \nu \left[a_6^c(\nu) + a_6^{\log}(\nu) \log u\right] u^6$$

$$A(u) = P_5^1[A_{\rm orb}^{\rm 5PN}(u)]$$

← here the logarithmic terms were kept **constant** when applying the Padé

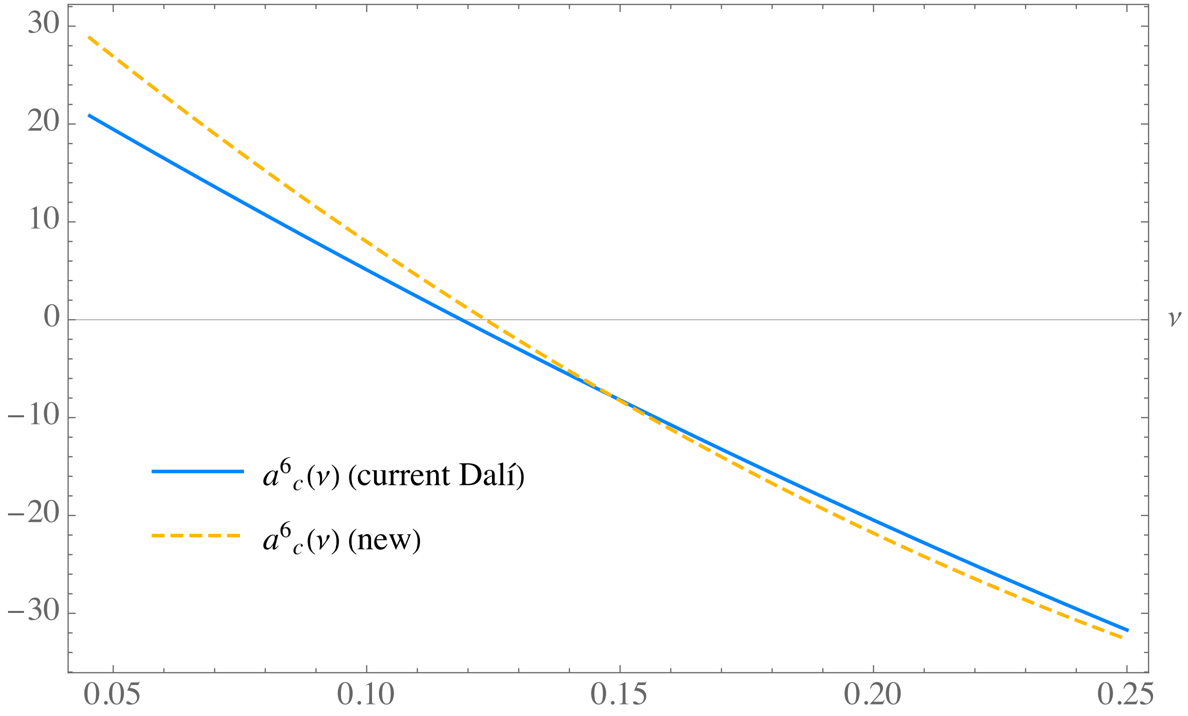
- New way of resumming (see arxiv:2407.04762):

$$A_{\rm orb}^{\rm PN}(u) \equiv A_{\rm poly}(u) + A_{\log}(u) \log u$$

$$A(u) = P_3^3[A_{\rm poly}] + a_5^{\log} u^5 P_1^0 \left[\frac{A_{\log}}{a_5^{\log} u^5}\right] \log u$$

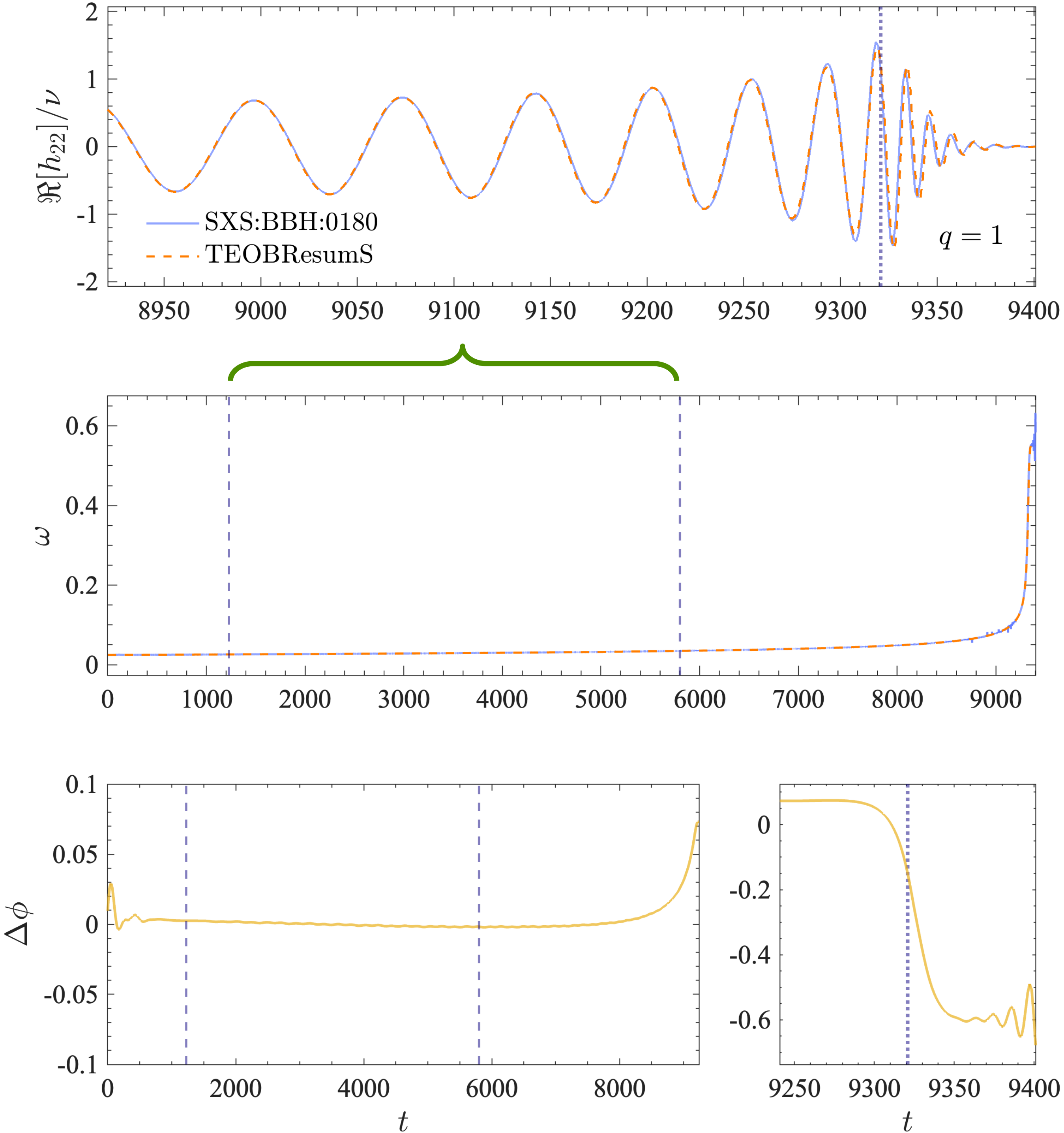
NEW VERSION OF THE LARGE-MASS-RATIO MODEL

- Dissipative sector: same
- **New potentials**:
  - new resummation
  - calibration of a^6_c exploiting latest SXS simulations for $q = \{1, 2, 4, 8, 20\}$
- Allow inclusion of the ringdown!



(preliminary results)

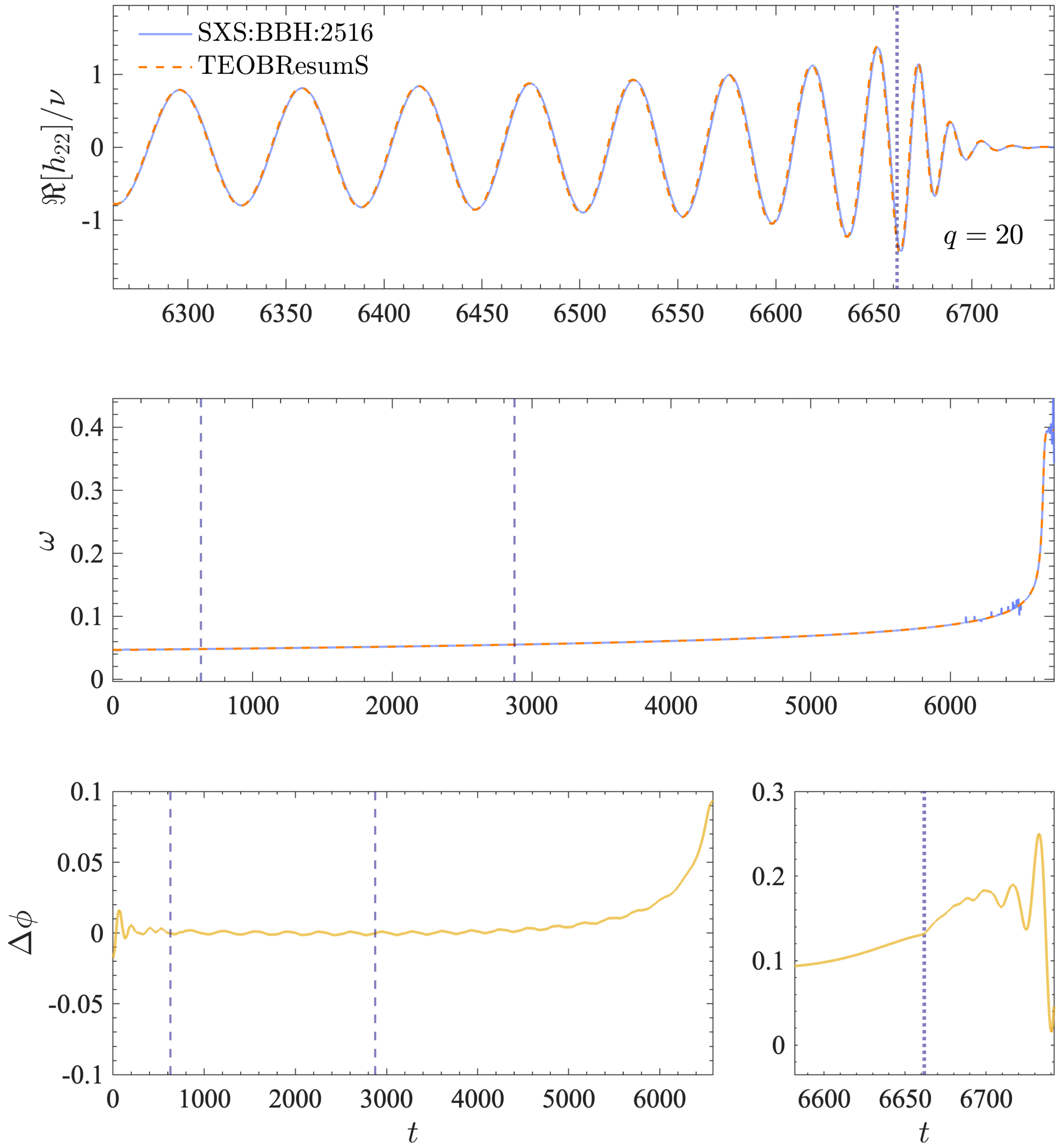
$$h^{\rm EOB}_{\ell m} = \theta(t_m - t) h^{\rm insplunge}_{\ell m}(t) + \theta(t - t_m) h^{\rm ringdown}_{\ell m}(t)$$



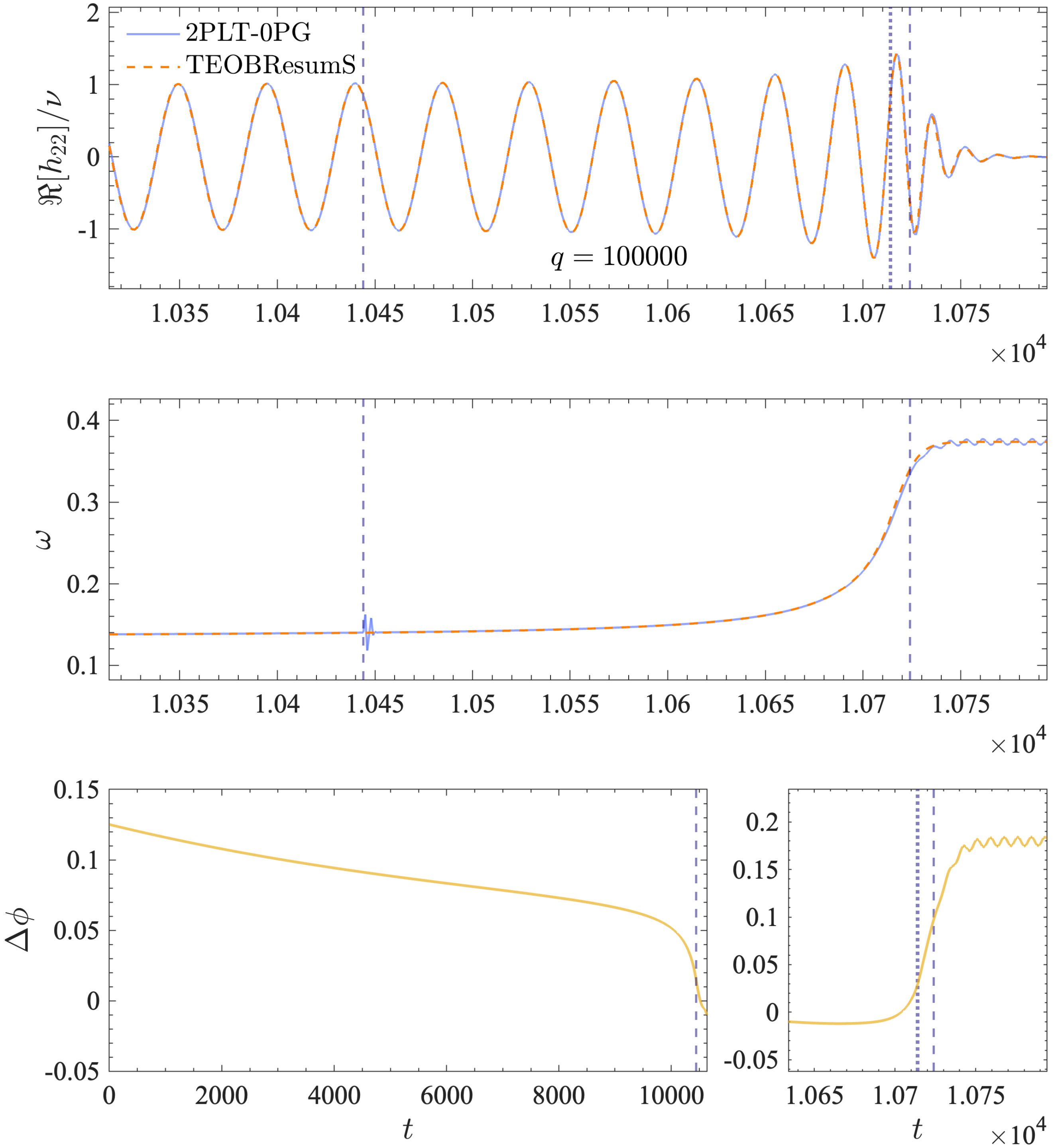
- Waveforms aligned by minimizing the root mean square of the phase difference on an interval
- Final dephasing:

$\Delta\phi \simeq -0.15$ rad

(even if we do not need a model for comparable masses :D)

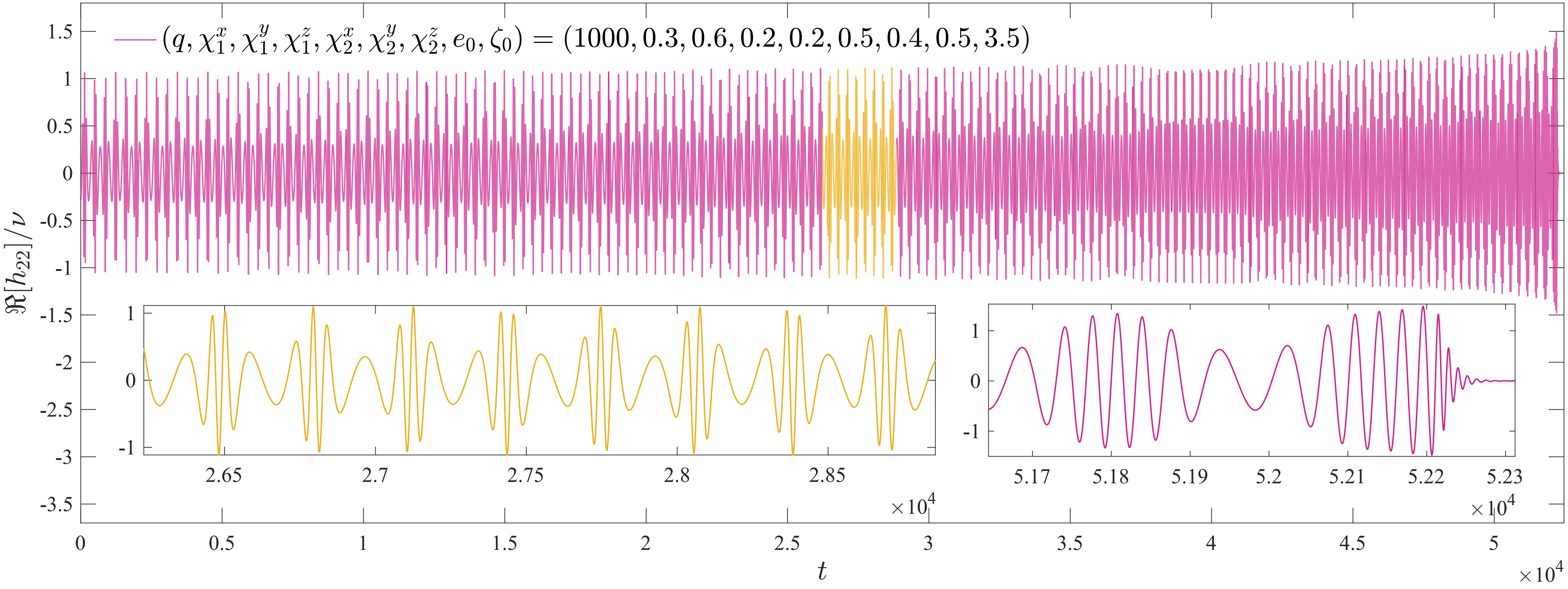


- SXS:BBH:2516 used for calibration
- Final dephasing: $\Delta\phi \simeq 0.13$ rad

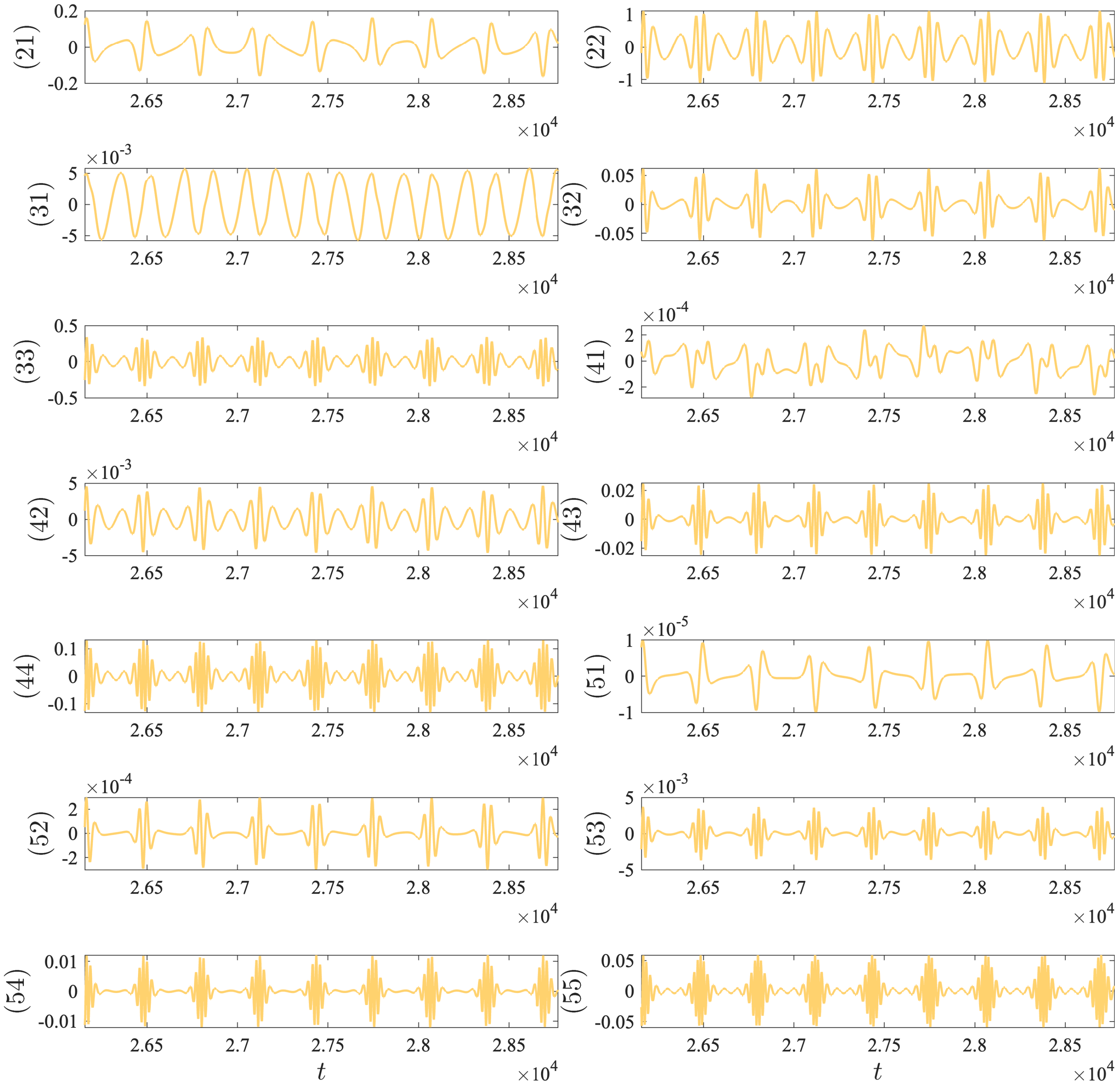


- GSF waveform: transition-to-plunge + geodesic plunge (work in progress by Lorenzo Küchler, Adam Pound & collaborators)
- Alignment interval chosen around plunge/merger
- Final dephasing: $\Delta\phi \simeq 0.03$ rad

BEYOND QUASICIRCULAR BINARIES



- TEOBResumS infrastructure: eccentricity & spin precession (arXiv:2404.15408, arXiv:2503.14580)
- Precession is based on a PN twist of the waveform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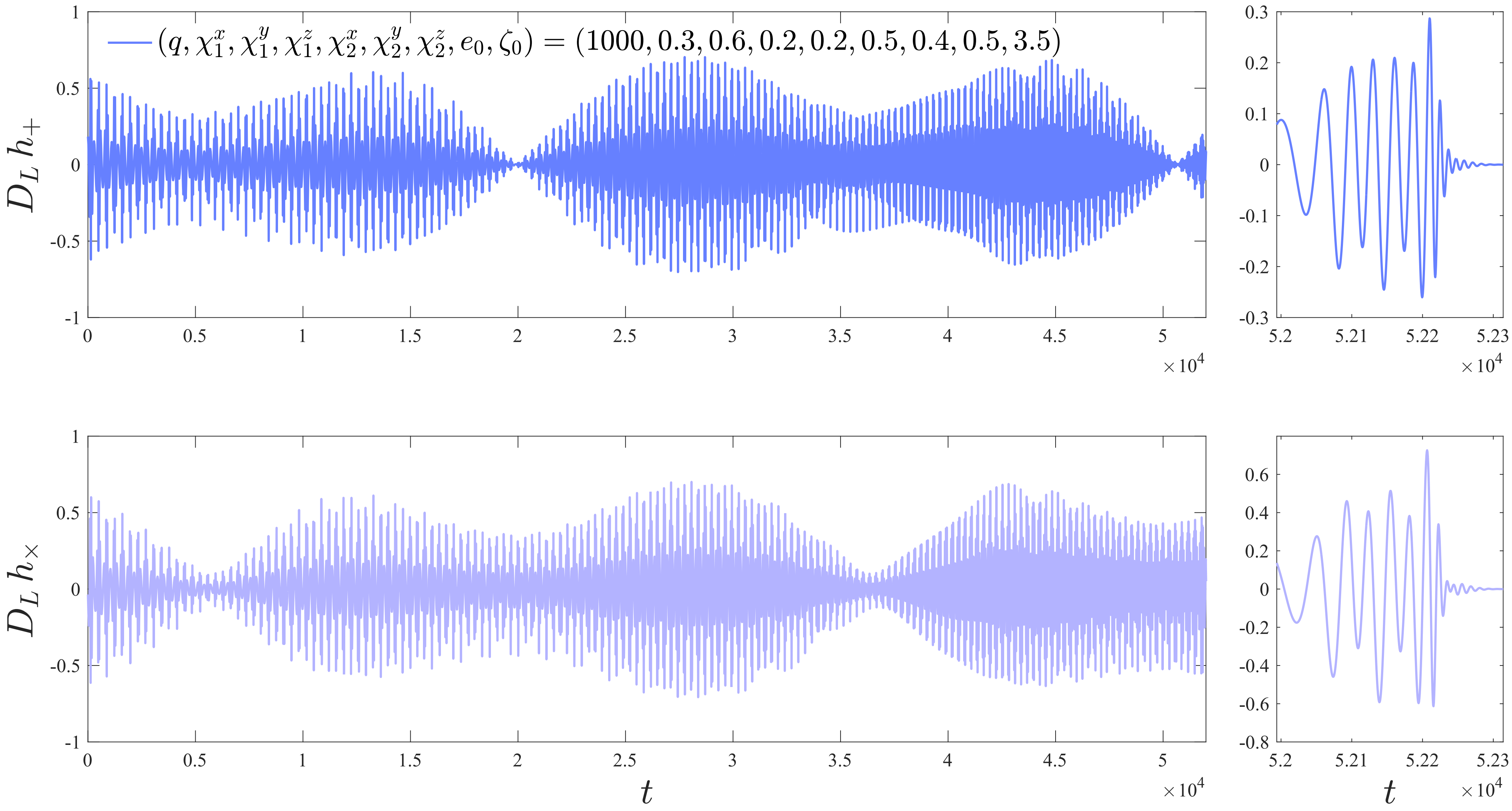


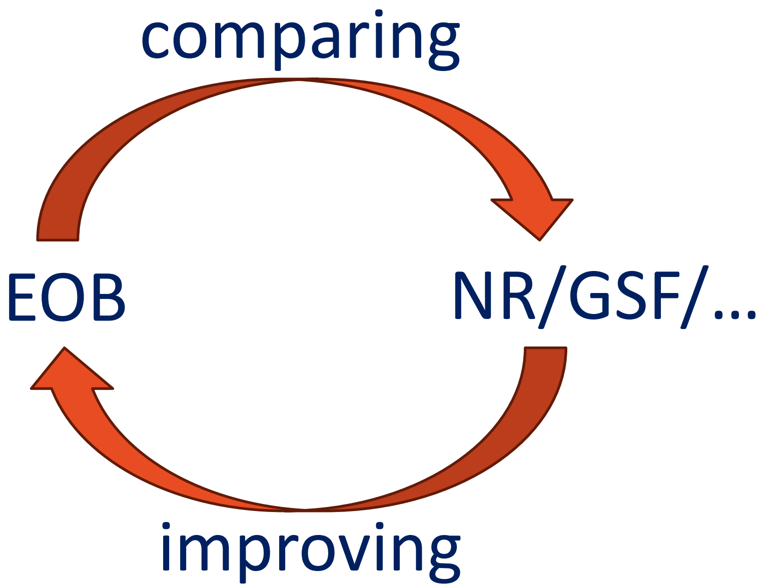
- Strain computed with 22, 21, 33, 44 modes

TO DO LIST

- Tests on various corners of the parameter space
- Improving the ringdown, the spin contribution to the flux
- Waveform acceleration!
- Environment, resonances, beyond GR…

FINAL REMARKS

- Now covering inspiral-merger-ringdown across different mass-ratio regimes (& fully generic configurations)
- Public code: https://bitbucket.org/teobresums/teobresums/src/Dali/ (branch: dev/DALI-rholm22PN)
- The cycle of growth:

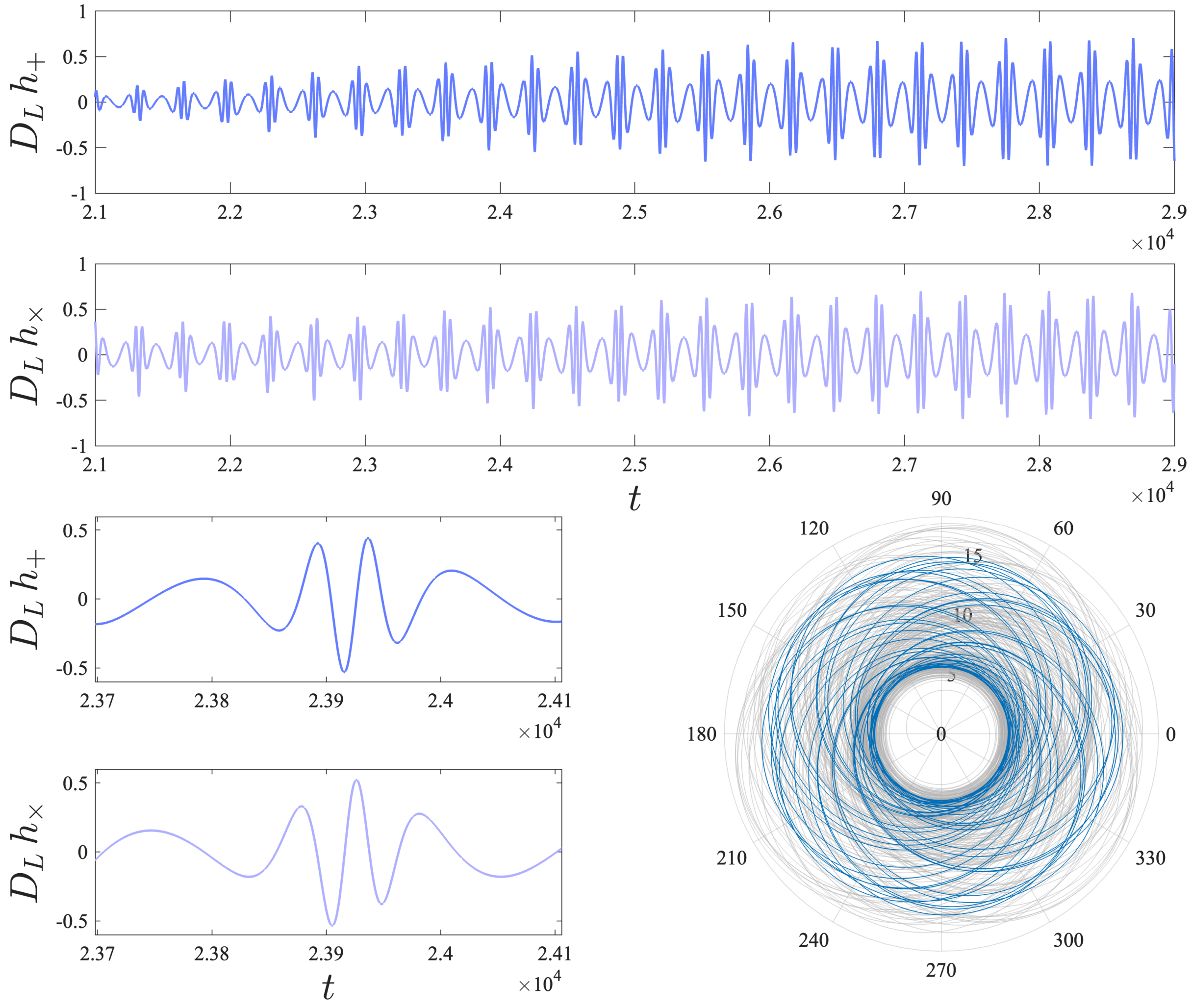


Thanks for your attention!

Questions? :)

Backup slides

SECTION & TRAJECTORY



DYNAMICAL BACKGROUND

$G = c = 1 \quad u = 1/r$

- Continuous deformation in ν of a Schwarzschild metric:

$$ds^2_{\text{eff}} = g^{\text{eff}}_{\mu\nu} dx^\mu_{\text{eff}} dx^\nu_{\text{eff}} = -A(r)dt^2 + B(r)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- EOB Hamiltonian for nonspinning binaries:

$$\hat{H}_{\text{EOB}} \equiv \frac{H_{\text{EOB}}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_{\text{eff}} - 1\right)} \qquad p_{r_*} = (A/B)^{1/2} p_r$$

$$\hat{H}_{\text{eff}} = \sqrt{p^2_{r_*} + A(u)\left(1 + p^2_\varphi u^2 + Q(u, p_{r_*})\right)}$$

- $A(u)$, $D(u) \equiv A(u)B(u)$ and $Q(u, p_{r*})$ are the three EOB potentials

DYNAMICS & WAVEFORM

G = c = 1

- Hamiltonian: $$\hat{H}_{\rm EOB} \equiv \frac{H_{\rm EOB}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}$$ $$p_{r_*} = (A/B)^{1/2} p_r$$

$$\hat{H}_{\rm eff} = \underbrace{\sqrt{p_{r_*}^2 + A\left(1 + \frac{p_\varphi^2}{r_c^2} + Q\right)}}_{\text{orbital}} + \underbrace{p_\varphi\left(G_S \hat{S} + G_{S_*}\hat{S}_*\right)}_{\text{spin-orbit}}$$

A, $D \equiv A\,B$ and Q are the three EOB potentials

- Hamiltonian equations of motion complemented by the radiation reaction:

$$\begin{cases} \dfrac{d\varphi}{dt} = \dfrac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi} = \Omega \\ \dfrac{dr}{dt} = \left(\dfrac{A}{B}\right)^{1/2} \dfrac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}} \\ \dfrac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi \\ \dfrac{dp_{r_*}}{dt} = -\left(\dfrac{A}{B}\right)^{1/2} \dfrac{\partial \hat{H}_{\rm EOB}}{\partial r} + \hat{\mathcal{F}}_r \end{cases}$$

- The multipoles are analytical functions of the phase space variables

$$h_+ - i h_\times = \frac{1}{\mathcal{D}_L} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m}\, {}_{-2}Y_{\ell m}$$

- The flux expressions are also analytical (evaluated at every t in the solution of the ODEs)

$$\mathcal{F}_{\ell m} \propto \left|\hat{h}_{\ell m}\right|^2$$

RADIATION REACTION

$$\dot{j}_{\text{system}} = \hat{\mathcal{F}}_\varphi = -\dot{j}_\infty - \dot{j}_{H_1} - \dot{j}_{H_2}$$

$$\hat{\mathcal{F}}_\varphi = \hat{\mathcal{F}}_\varphi^\infty + \hat{\mathcal{F}}_\varphi^H$$

asymptotic contribution — horizon contribution

$$\hat{\mathcal{F}}_\varphi^\infty = -\frac{32}{5}\nu\, r_\omega^4 \Omega^5 \hat{f}^\infty(v_\varphi^2;\nu)$$

$$1 = \Omega^2 r_\omega^3$$

Modified Kepler's law valid during the plunge

Reduced flux function: $\hat{f}^\infty = \frac{1}{\mathcal{F}_{22}^{\text{Newt}}} \sum_{\ell m} \mathcal{F}_{\ell m}$

energy flux radiated at infinity

$$\mathcal{F}_{\ell m} = \mathcal{F}_{\ell m}^{\text{Newt}} \left|\hat{h}_{\ell m}\right|^2$$

correction from the waveform

WAVEFORM: STRUCTURE AND CONVENTIONS

- Strain: $$h_+ - ih_\times = \frac{1}{\mathscr{D}_L} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} \, {}_{-2}Y_{\ell m}$$

Multipoles: computed with the phase space variables found by solving the Hamiltonian equations of motion

Newtonian prefactor

Resummed PN correction

$$h_{\ell m} = h_{\ell m}^{(N,\epsilon)} \, \hat{h}_{\ell m} = h_{\ell m}^{(N,\epsilon)}(x) \, \hat{S}_{\rm eff}^{\epsilon}(x) \, \hat{h}_{\ell m}^{\rm tail}(x) \, \left[\rho_{\ell m}(x)\right]^{\ell}$$

$$x = v_\varphi^2 \equiv (r_\omega \Omega)^2$$

- Regge-Wheeler-Zerilli normalized waveform:

$$\Psi_{\ell m} \equiv \frac{h_{\ell m}}{\sqrt{(l+2)(l+1)l(l-1)}} = A_{\ell m} e^{-i\phi_{\ell m}}$$

waveform frequency

$$\omega_{\ell m} = \dot{\phi}_{\ell m}$$

SPINNING BINARIES

Dimensionless spins $\chi_i = \frac{S_i}{M_i^2}, \quad i = 1,2$

$$X_1 = \frac{m_1}{M} = \frac{1}{2}\left(1 + \sqrt{1 - 4\nu}\right)$$

$$X_2 = \frac{m_2}{M} = 1 - X_1 \qquad \tilde{a}_i \equiv \chi_i X_i$$

Dimensionless effective Kerr parameter: $\tilde{a}_0 = \tilde{a}_1 + \tilde{a}_2 = \chi_1 X_1 + \chi_2 X_2$

The effective Hamiltonian has orbital + spin-orbit contribution:

$$\hat{H}_{\rm eff} = \hat{H}_{\rm eff}^{\rm orb} + p_\phi \left(G_S \hat{S} + G_{S_*} \hat{S}_*\right)$$

gyro-gravitomagnetic functions

$$S = S_1 + S_2 \qquad \hat{S} \equiv \frac{S}{M^2}$$

$$S_* = \frac{M_2}{M_1} S_1 + \frac{M_1}{M_2} S_2 \qquad \hat{S}_* \equiv \frac{S_*}{M^2}$$

The orbital Hamiltonian is a function of the centrifugal radius:

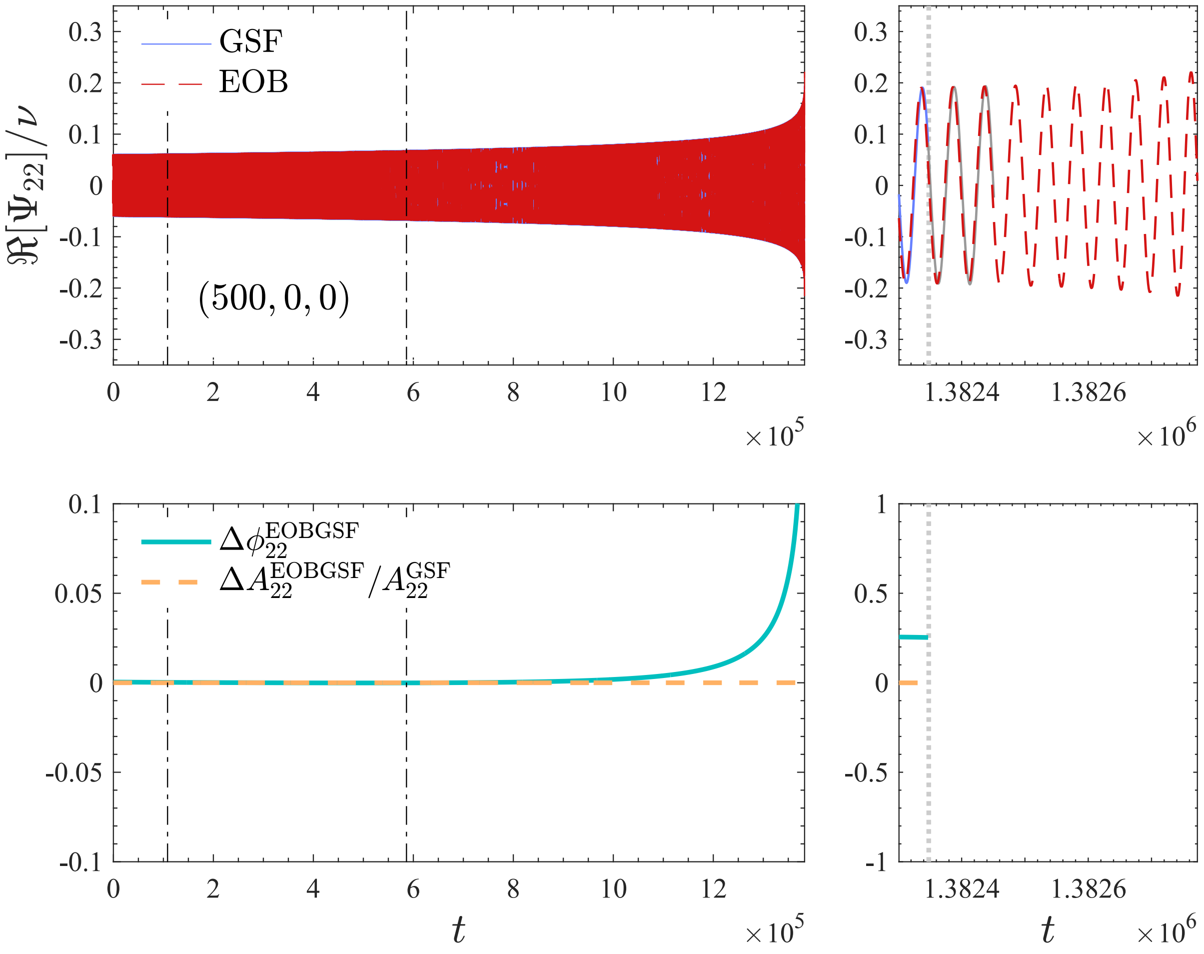
$$r_c^2 = r^2 + \tilde{a}_0^2 \left(1 + \frac{2}{r}\right) + \delta\hat{a}^2$$

spin-spin contribution

STANDARD PRACTICE: TIME-DOMAIN PHASING

GSF: $\Psi^1_{22} = A^1_{22}(t_1)e^{-i\phi_1(t_1)}$

EOB: $\Psi^2_{22} = A^2_{22}(t_2-\tau)e^{-i[\phi_2(t_2-\tau)-\alpha]}$



- We focus on the $\ell = m = 2$ strain multipole
- **Phasing**: finding the time and phase shift ($\boldsymbol{\tau}$, α) by minimizing the root-mean-square of the phase difference on a chosen interval

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Delta\phi(t_i,\tau,\alpha))^2}$$

$$\Delta\phi(t_i,\tau,\alpha) = \big(\phi_2(t_i-\tau)-\alpha\big)-\phi_1(t_i)$$

GSF-INFORMED EOB POTENTIALS

$$A(u;\nu) = 1 - 2u + \nu a_{1\mathrm{SF}}(u) + O(\nu^2)$$

$$\bar{D}(u;\nu) = \frac{1}{AB} = 1 + \nu \bar{d}_{1\mathrm{SF}}(u) + O(\nu^2)$$

$$Q(u, p_{r_*};\nu) = \nu q_{1\mathrm{SF}}(u) p_{r_*}^4 + O(\nu^2)$$

New EOB orbital potentials
(the standard choice was 5PN for A and 3PN for D and Q, with all the available info in ν, Padé-resummed…)

Expressions for a_{1SF} , $\bar{d}_{1SF}$, q_{1SF} at 8.5PN order
+ suitable factorization & Padé-resummation
+ fit on the numerical GSF data of
Akcay & van de Meent, Phys. Rev. D 93, 064063 (2016)
(see arXiv:2207.14002v1)

Side note: these potentials are singular at the light ring!

RADIATION REACTION PT. 1 (FLUX AT ∞)

- Flux multipoles are factorized into different contributions, among which the residual amplitude corrections:

memo

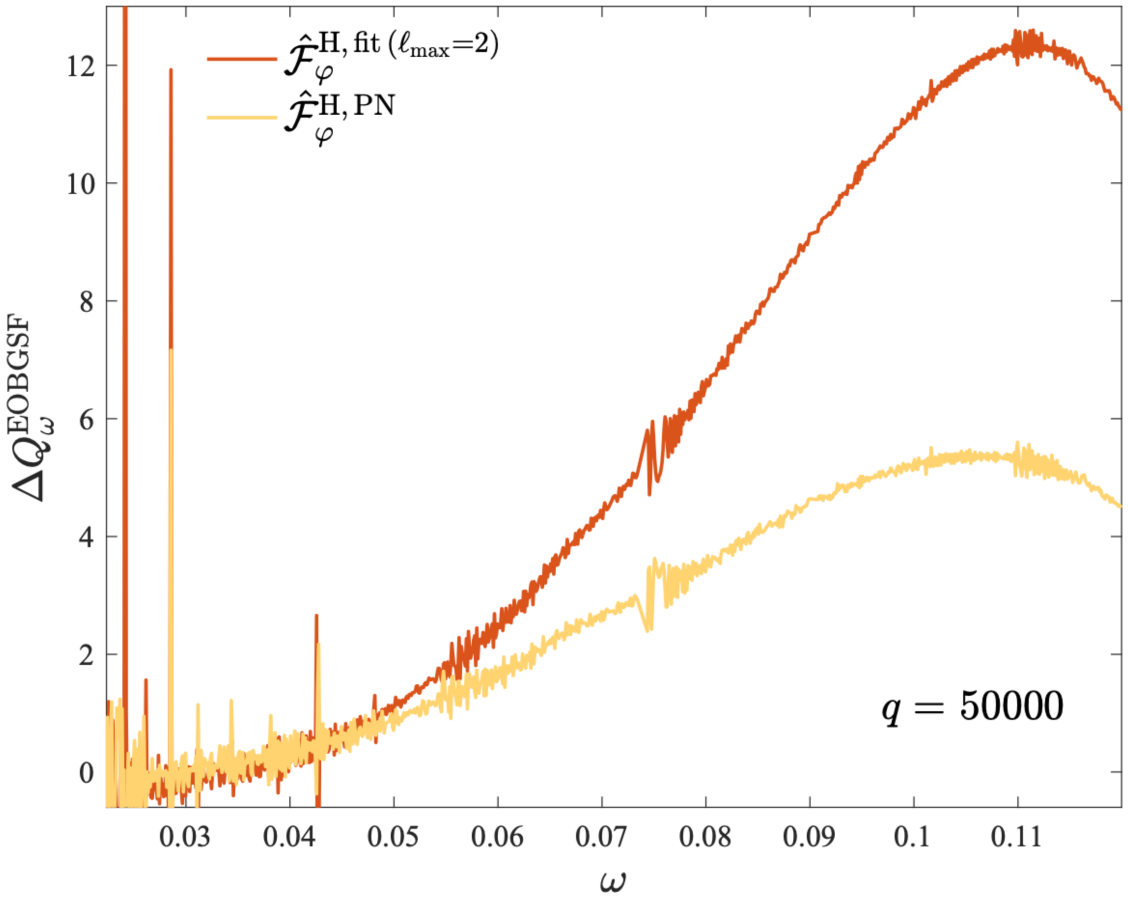
$$\mathcal{F}_{\ell m} = \mathcal{F}^{\rm Newt}_{\ell m} \left|\hat{h}_{\ell m}\right|^2$$

$$\hat{h}_{\ell m} = \hat{S}^{\epsilon}_{\rm eff}\,\hat{h}^{\rm tail}_{\ell m}\,\left(\rho_{\ell m}\right)^{\ell}$$

$$\rho_{\ell m} = \underbrace{1 + c_1 x + c_2 x^2 + \dots}_{\text{PN series}}$$

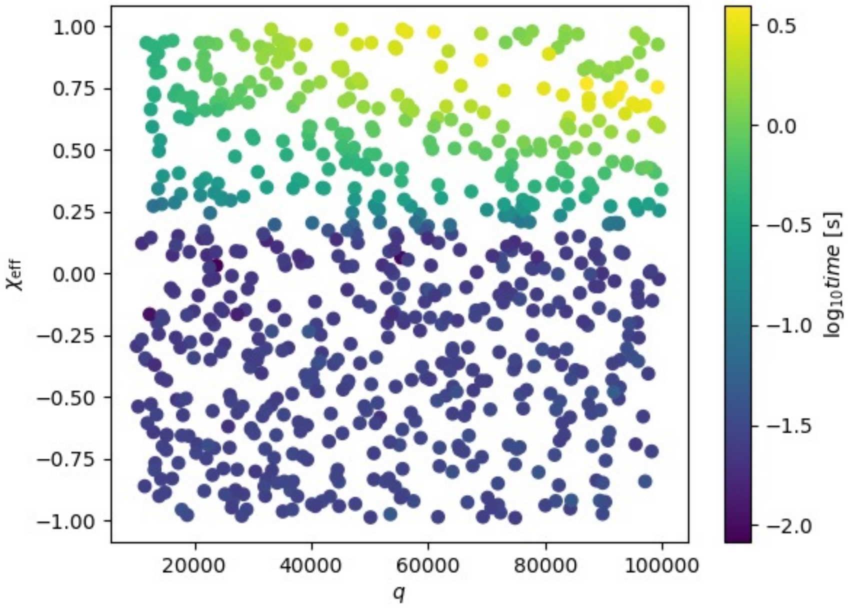
- The standard TEOBResumS had Padé-resummed 6PN expressions with the available info in ν up to 3PN, i.e. $c_1(\nu), c_2(\nu), c_3(\nu)$, hence dubbed 3^{+3}PN
- Now we use 22PN results (Fujita 2012), non resummed, with the same info in ν (hence 3^{+19}PN)

RADIATION REACTION PT. 2 (HORIZON FLUX)

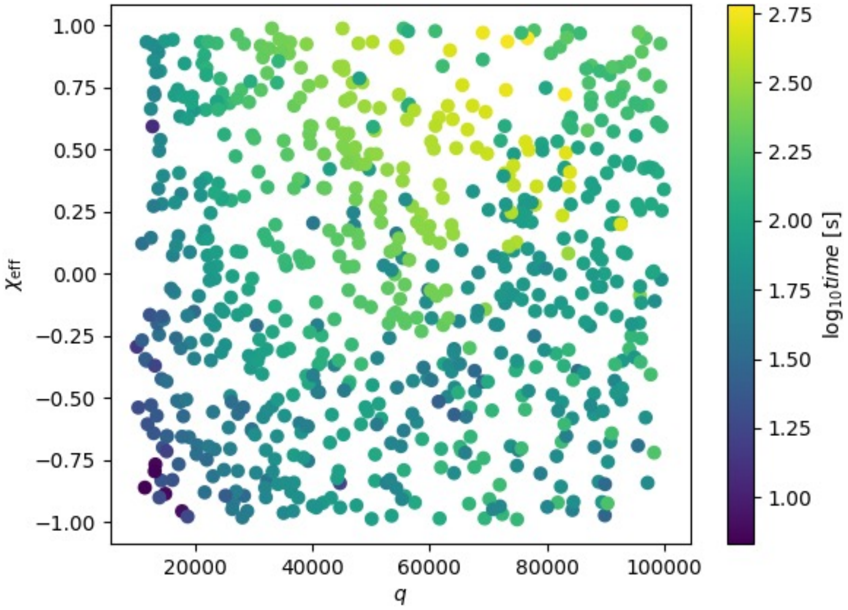


- The standard horizon flux in TEOBResumS had only $\ell = 2$ multipoles
- We implemented a new 'hybrid' version with multipoles up to $\ell = 4$, where some of the multipoles are PN-expanded expressions, other ones are a fit to numerical data (arXiv:1207.0769v2)
- Halves the dephasing!

OUR CURRENT PACE



with post-adiabatic analytical solution to the equations of motion in the inspiral



without
(full solution of the ODEs)