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Teukolsky-Compatible Symmetry Operators for Maxwell Fields

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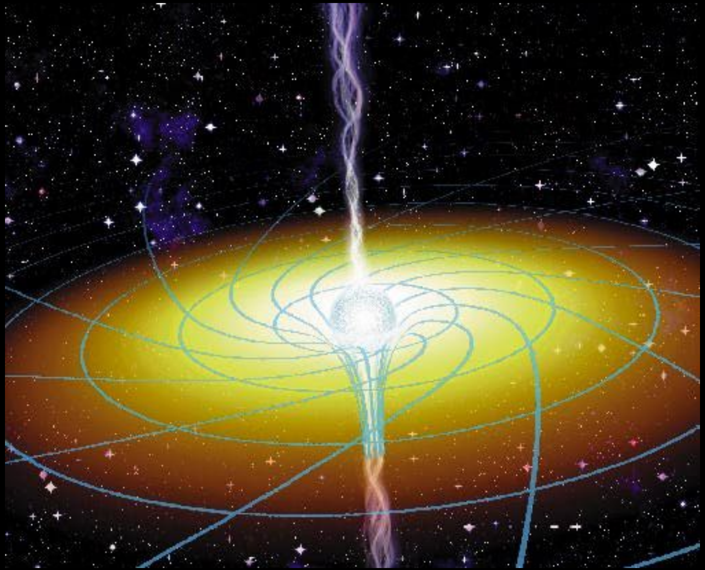
Motivation

Why do we decouple and separate field equations?

To extract information contained in astrophysical phenomena

Kerr Black Hole *(t, r, θ, ϕ)*



Accretion disks, relativistic jets, and gravitational-wave emission

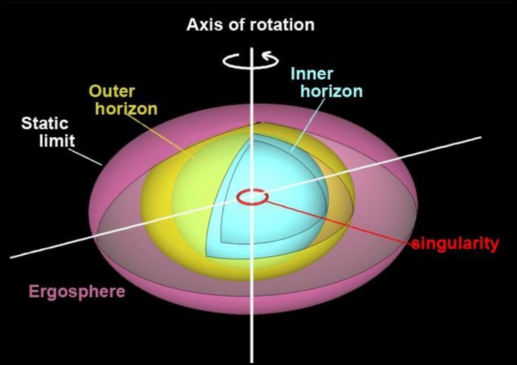
↓

Coupled PDEs

↓

DIFFICULT TO SOLVE DIRECTLY!

Kerr spacetime hidden symmetries

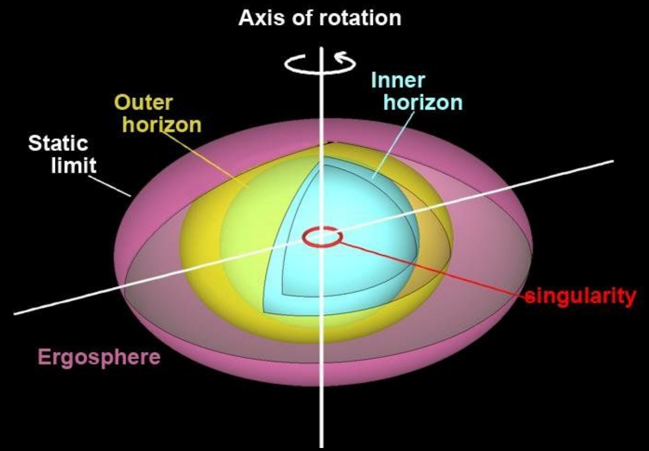


→ **Decouple/separate field equations**

The 'superhero' of rotating black hole physics

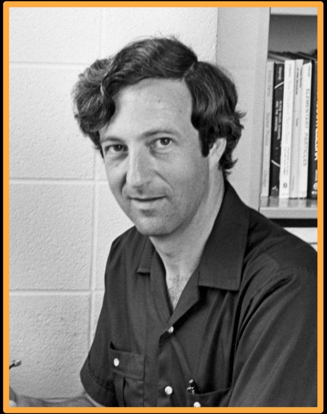
Principal Killing-Yano tensor (PKY) encodes a hidden symmetry which ensures:

- ➔ Complete *integrability* of geodesic motion
- ➔ *Separability* of various field equations *(Hamilton-Jacobi, Klein-Gordon, Dirac, Teukolsky)*



1973: All massless-field perturbations (spin s) of Kerr:

$$\psi_s = \begin{cases} \Phi, & s = 0 \text{ (scalar)} \\ \phi_0 = F_{\mu\nu}\, l^\mu m^\nu \quad , \quad \phi_2 = F_{\mu\nu}\, \bar{m}^\mu n^\nu, & s = \pm 1 \text{ (electromagnetic)} \\ \Psi_0 \ , \ \Psi_4, & s = \pm 2 \text{ (gravitational)} \end{cases}$$



S. A. Teukolsky

satisfy a single "**master**" **equation** (PDE):

$$\boxed{\mathcal{T}_s\left[\psi_s\right] \ = \ 0}$$

Then, the **separation ansatz:**

$$\boxed{\psi_s(t, r, \theta, \phi) = e^{-i\omega t + im\phi}\, R_{lm\omega}(r)\, S_{lm\omega}(\theta)\,,}$$

Splits the **PDE** into two **radial** and **angular** **ODE's** connected by a **separation constant** $\boxed{\lambda_{\ell m s}}$

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S. A. Teukolsky

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Symmetries → **Symmetry Operators** → **Separability**

explicit / hidden

↓

Construct a set of mutually commuting symmetry operators:

$$[i\partial_t \ , \mathcal{T}_s] = [i\partial_\phi \ , \mathcal{T}_s] = [\mathcal{K}_s \ , \mathcal{T}_s] = 0$$

↓

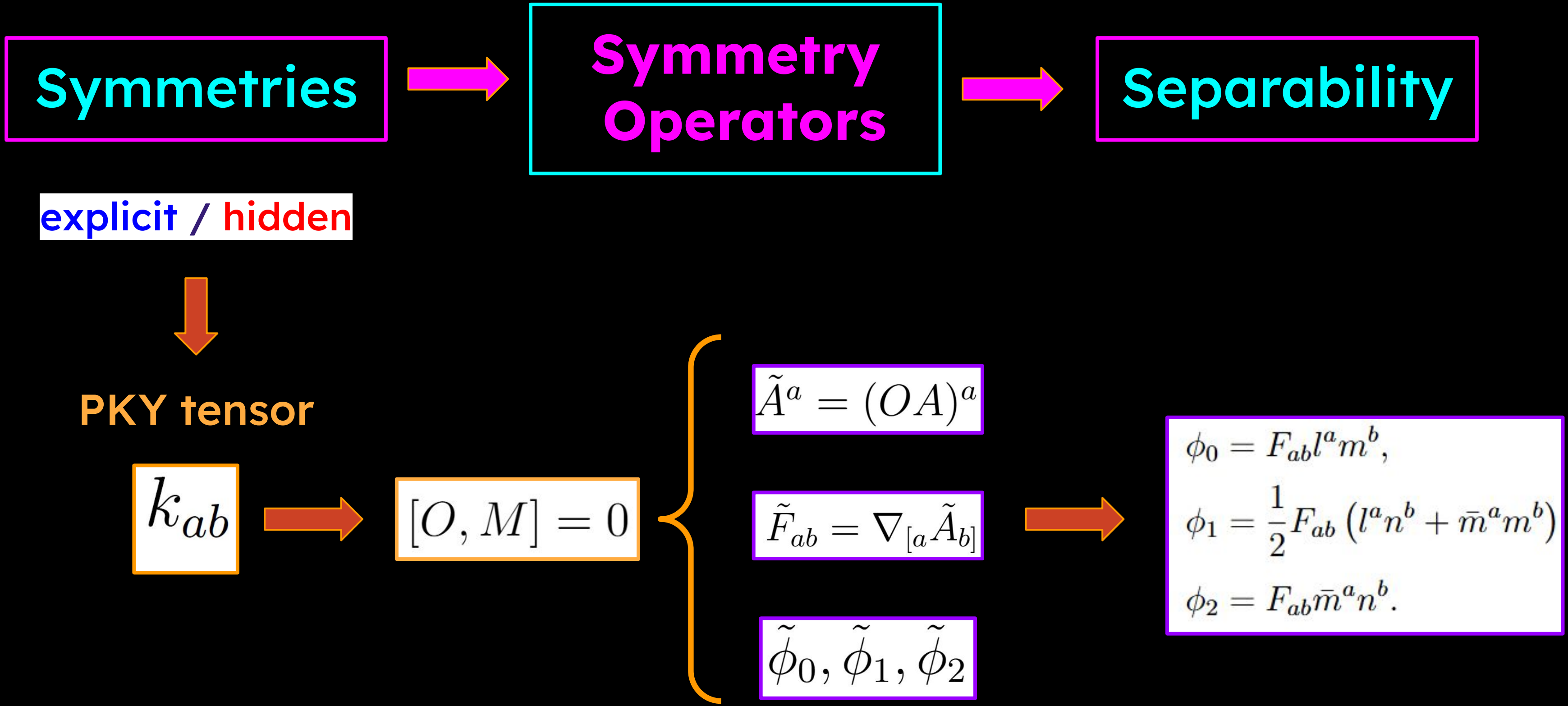
Choose a common eigenfunction:

$$i\partial_t\psi_s = \omega\psi_s,$$

$$i\partial_\phi\psi_s = m\psi_s,$$

$$\mathcal{K}_s\psi_s = \lambda\psi_s,$$

$$\mathcal{T}_s\psi_s = 0$$



Symmetries

Symmetry Operators

Separability

explicit / hidden

PKY tensor

$$\boxed{k_{ab}}$$

$$\boxed{[O, M] = 0}$$

$$\boxed{\tilde{A}^a = (OA)^a}$$

$$\boxed{\tilde{F}_{ab} = \nabla_{[a}\tilde{A}_{b]}}$$

$$\boxed{\tilde{\phi}_0, \tilde{\phi}_1, \tilde{\phi}_2}$$

$$\phi_0 = F_{ab} l^a m^b,$$
$$\phi_1 = \frac{1}{2} F_{ab} \left(l^a n^b + \bar{m}^a m^b \right)$$
$$\phi_2 = F_{ab} \bar{m}^a n^b.$$

The rest of the talk...

- *Kerr geometry and NP/GHP formalism review*
- *Symmetry operators from the PKY tensor*
- *Examples: Action on fields*
- *Final remarks*

NP/GHP formalism

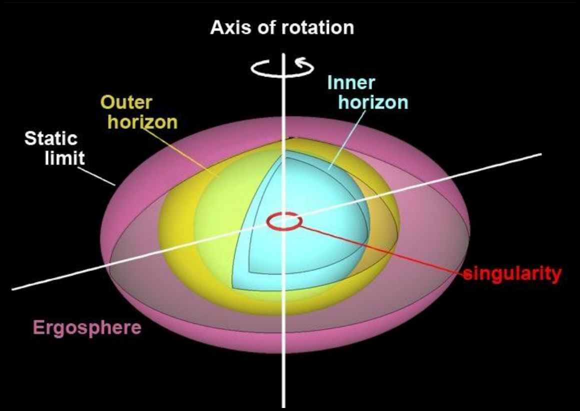
Kerr black hole

In Boyer-Lindquist coordinates:

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2,$$

with:

$$\Delta = r^2 - 2Mr + a^2 = (r - r_p)(r - r_m), \quad \Sigma = r^2 + a^2\cos^2\theta.$$



Kinnersley tetrad:

$$\{l^a, n^a, m^a, \bar{m}^a\}$$

$$l^a = \frac{1}{\sqrt{2}\Delta}\left[\left(r^2+a^2\right)\boldsymbol{\partial_t} + \Delta\boldsymbol{\partial_r} + a\boldsymbol{\partial_\phi}\right],$$

$$n^a = \frac{1}{\sqrt{2}\Sigma}\left[\left(r^2+a^2\right)\boldsymbol{\partial_t} - \Delta\boldsymbol{\partial_r} + a\boldsymbol{\partial_\phi}\right],$$

$$m^a = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta\boldsymbol{\partial_t} + \boldsymbol{\partial_\theta} + i\csc\theta\boldsymbol{\partial_\phi}\right).$$

Newman-Penrose (NP) formalism

Tensor components into smaller set of complex scalars.

- **Null tetrad** basis: $\{l^a, n^a, m^a, \bar{m}^a\}$
- Directional **derivatives**

$$D \equiv l^a \nabla_a, \quad \Delta \equiv n^a \nabla_a, \quad \delta \equiv m^a \nabla_a, \quad \bar{\delta} \equiv \bar{m}^a \nabla_a$$

- **Spin coefficients** (connection)

$$\kappa, \quad \sigma, \quad \lambda, \quad \nu, \quad \rho, \quad \mu, \quad \tau, \quad \pi, \quad \epsilon, \quad \gamma, \quad \alpha, \quad \beta.$$

- Curvature tensors, EFE, Bianchi identities, etc

Electromagnetic fields (s = ± 1)

$$\nabla^a F_{ab} = \frac{1}{2}\nabla^a\nabla_{[a}A_{b]} = 0,$$

NP form

NP coupled Maxwell equations

$$(D - 2\rho)\phi_1 - (\bar{\delta} + \pi - 2\alpha)\phi_0 = 0$$

$$(\Delta + 2\mu)\phi_1 - (\delta - \tau + 2\beta)\phi_2 = 0$$

$$(D - \rho + 2\epsilon)\phi_2 - (\bar{\delta} + 2\pi)\phi_1 = 0$$

$$(\Delta + \mu - 2\gamma)\phi_0 - (\delta - 2\tau)\phi_1 = 0$$

Maxwell scalars

$$\phi_0 = F_{ab}l^a m^b,$$

$$\phi_1 = \frac{1}{2}F_{ab}\left(l^a n^b + \bar{m}^a m^b\right)$$

$$\phi_2 = F_{ab}\bar{m}^a n^b.$$

satisfy

$$\begin{aligned}&[(D - \epsilon + \bar{\epsilon} - 2\rho - \bar{\rho})(\Delta - 2\gamma + \mu)\\ &\quad - (\delta + \bar{\pi} - \bar{\alpha} - \beta - 2\tau)(\bar{\delta} + \pi - 2\alpha)\,]\,\phi_0 = 0,\end{aligned}$$

$$\begin{aligned}&[(\Delta + 3\gamma - \bar{\gamma} - 2\mu + \bar{\mu})(D - 2\epsilon - \rho)\\ &\quad - (\bar{\delta} - \bar{\tau} + \bar{\beta} - 3\alpha - 2\pi)(\delta - \tau - 2\beta) - 6\psi_2]\,\phi_2 = 0,\end{aligned}$$

Decoupled Teukolsky master equations

Geroch-Held-Penrose (GHP) *formalism*

NP equations are *not covariant* under spin/boost rescaling of the null vectors.

- **Same null tetrad** plus spin/boosts **weights *(p, q).***
- ***Under:***

$$\boxed{(\ell, n, m, \bar{m}) \ \mapsto \ (\underbrace{A\,\ell,\ A^{-1}n}_{\text{boost}},\ \underbrace{e^{i\theta}m,\ e^{-i\theta}\bar{m}}_{\text{rotation}}),}$$

*a **GHP scalar transforms** as:* $\boxed{\eta \ \rightarrow \ A^{p}e^{iq\theta}\,\eta}$

- ***GHP operators***

$$\boxed{\begin{aligned} \text{þ}\eta &= (D - p\epsilon - q\bar{\epsilon})\eta, & \text{þ}'\eta &= (\Delta + p\epsilon' + q\bar{\epsilon}')\eta, \\ \text{ð}\eta &= (\delta - p\beta + q\bar{\beta}')\eta, & \text{ð}'\eta &= (\bar{\delta} + p\beta' - q\bar{\beta})\eta. \end{aligned}}$$

Maxwell Equations (ME) *with its adapted tetrad written in the GHP formalism read:*

$$\begin{aligned} \text{þ}\Phi_1 &= (\eth' + \tau')\Phi_0\,, \\ \text{þ}'\Phi_1 &= (\eth + \tau)\Phi_0\,, \end{aligned}$$

$$\begin{aligned} \eth\Phi_1 &= (\text{þ}' + \rho')\Phi_0\,, \\ \eth'\Phi_1 &= (\text{þ} + \rho)\Phi_0\,, \end{aligned}$$

where $\Phi_j = \kappa_1^2\phi_j$

Applying the GHP operators on the ME we get the **Teukolsky - Starobinsky Identities (TSI):**

$$\begin{aligned} \text{þ}'\text{þ}'\Phi_0 &= \eth\eth\Phi_2\,, \\ \eth'\eth'\Phi_0 &= \text{þ}\text{þ}\Phi_2\,, \\ \left(\eth'\text{þ}' + \bar{\rho}'\eth'\right)\Phi_0 &= \left(\eth\text{þ} + \bar{\rho}\eth\right)\Phi_2, \end{aligned}$$

Finally, **Teukolsky Master Equations (TME)** *in GHP form:*

$$\begin{aligned} \mathbb{T}_0[\phi_0] &\equiv \left[(\text{þ} - \bar{\rho})\,(\text{þ}' + \rho') - (\eth - \bar{\tau}')\,(\eth' + \tau')\right]\kappa_1^2\phi_0 = 0\,, \\ \mathbb{T}_1[\phi_1] &\equiv \left[(\text{þ}' + \rho' - \bar{\rho}')\,\text{þ} - (\eth' - \bar{\tau} + \tau')\,\eth\right]\kappa_1^2\phi_1 = 0\,, \\ \mathbb{T}_2[\phi_2] &\equiv \left[(\text{þ}' - \bar{\rho}')\,(\text{þ} + \rho) - (\eth' - \bar{\tau})\,(\eth + \tau)\right]\kappa_1^2\phi_2 = 0. \end{aligned}$$

Symmetry operators from the PKY tensor

Principal Killing-Yano (PKY) tensor

Principal tensor = a (non-degenerate) closed conformal Killing Yano 2-form.

The PKY tensor satisfies the following defining equation

$$\nabla_c k_{ab} = g_{ca}\xi_b - g_{cb}\xi_a .$$

Frolov, V.P., Krtouš, P. & Kubizňák, D., *Living Rev Relativ* (2017) 20: 6.

Homogeneous symmetry operators for vector perturbations

Linear in *k*: →

$$\boxed{O^{(v)}_{k} = k^{a}{}_{b}\nabla^2 - k_{cb}\nabla^c\nabla^a - k^{ac}\nabla_c\nabla_b - 2\xi_b\nabla^a .}$$

Quadratic in *k*:

$$\boxed{O^{(v)}_{q1} = k^2\nabla^a\nabla_b + 4k^{a}{}_{c}\xi^c\nabla_b = \nabla^a\left(k^2\nabla_b\right) .}$$

$$\boxed{\begin{aligned} O^{(v)}_{q2} = {} & k^{a}{}_{c}k_{bd}\nabla^c\nabla^d + \nabla^2_{k^2}\delta^a_b - \left(k^2\right)^{a}{}_{e}\nabla_b\nabla^e - \left(k^2\right)_{be}\nabla^e\nabla^a \\ & + \frac{1}{2}k^2\nabla^2\delta^a_b - \frac{1}{2}k^2\nabla_b\nabla^a + \left(k^2\right)^{a}{}_{b}\nabla^2 - k^{a}{}_{b}\nabla_\xi \\ & - k^{e}{}_{f}\xi^f\nabla_e\delta^a_b - 2\xi^a k_{be}\nabla^e + \xi_b k^{a}{}_{e}\nabla^e - k_{eb}\xi^e\nabla^a . \end{aligned}}$$

$$\boxed{\begin{aligned} O^{(v)}_{q3} = {} & k^{a}{}_{b}\nabla^2_k - k^{a}{}_{c}k_{bd}\nabla^c\nabla^d + \left(k^2\right)^{a}{}_{e}\nabla^e\nabla_b + \left(k^2\right)_{be}\nabla^a\nabla^e \\ & - \left(k^2\right)^{a}{}_{b}\nabla^2 - 3k^{a}{}_{b}\nabla_\xi - k^{e}{}_{f}\xi^f\nabla_e\delta^a_b + \xi_b k^{a}{}_{e}\nabla^e \\ & + 3k_{eb}\xi^e\nabla^a - 2\xi^2\delta^a_b + 2\xi^a\xi_b . \end{aligned}}$$

$$\begin{aligned} \left(k^2\right)_{ab} &= k_{ac}k^{c}{}_{b}, \\ k^2 &= k_{ab}k^{ab}, \\ \nabla_\xi &= \xi^a\nabla_a, \\ \nabla^2_{k^2} &= \left(k^2\right)^{ab}\nabla_a\nabla_b . \end{aligned}$$

F Gray, D Kubizňák - Physical Review D, 2024

Constructing new operators in NP/GHP formalism

1. Act the operator on a four-potential

$$\tilde{A}^a = (OA)^a$$

2. Algebraic simplification

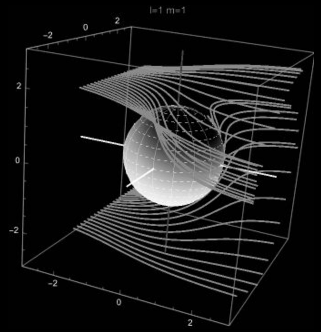
3. Construct the field strength

$$\tilde{F}_{ab} = \nabla_{[a}\tilde{A}_{b]}$$

4. Project into the null tetrad

$$\tilde{\phi}_0, \tilde{\phi}_1, \tilde{\phi}_2$$

5. *Apply to a given field*



Linear in k:

1.
$$\tilde{A}^a = \left(k^a{}_b \nabla^j \nabla_j - k_{cb} \nabla^c \nabla^a - k^{ac} \nabla_c \nabla_b - 2\xi_b \nabla^a\right) A^b.$$

2. Using source-free Maxwell equations, commutation of covariant derivatives, and the KY equation:

$$\tilde{A}_n = -k^{bc} \nabla_n F_{bc} + A^a k^{cd} R_{acnd} - 2\xi^a \nabla_n A_a.$$

3. *Field strength:*

$$\tilde{F}_{fn} = \nabla_f \left(F_{nb} \xi^b\right) - \nabla_n \left(F_{fb} \xi^b\right).$$

4. *Lie Derivative interpretation*

$$\tilde{F}_{ab} = -\mathcal{L}_\xi F_{ab}.$$

Quadratic operators

Linear combination
"minus":

$$\boxed{\tilde{A}^{-}_{a} = \tilde{A}^{(q_2)}_{a} - \tilde{A}^{(q_3)}_{a}}$$

$$\tilde{A}^{-}_{a} = (k,F)\xi_a - F_b{}^{c}k_{ac}\xi^b + 3F_a{}^{c}h_{bc}\xi^b - \nabla_b(k\cdot k\cdot F)^{b}{}_{a} - k_{ab}\nabla^b(F\cdot k).$$

Its field strength then takes the form:

$$\begin{aligned}\tilde{F}^{-}_{da} = {}& 4F_{bd}\xi_a\xi^b - 6F_{ad}\xi_b\xi^b + 4F_{ab}\xi^b\xi_d + k_d{}^{c}\xi^b(\nabla_a F_{bc}) - 3k_b{}^{c}\xi^b(\nabla_a F_{dc}) - 2\xi_d\nabla_a(k,F)\\ & - 3F_{dc}k_b{}^{c}(\nabla_a\xi^b) + F_{bc}k_d{}^{c}(\nabla_a\xi^b) - (k,F)(\nabla_a\xi_d) + \nabla_a\nabla_b(k,k,F)^{b}{}_{d} + k_d{}^{b}\nabla_a\nabla_b(k,F)\\ & + 3k_b{}^{c}\xi^b(\nabla_d F_{ac}) - 3k_a{}^{c}\xi^b(\nabla_d F_{bc}) + 2\xi_a\nabla_d(k,F) + (k,F)(\nabla_d\xi_a) - F_{bc}k_a{}^{c}(\nabla_d\xi^b)\\ & + 3F_{ac}k_b{}^{c}(\nabla_d\xi^b) - \nabla_d\nabla_b(k,k,F)^{b}{}_{a} - k_a{}^{b}\nabla_d\nabla_b(k,F).\end{aligned}$$

Quadratic operators

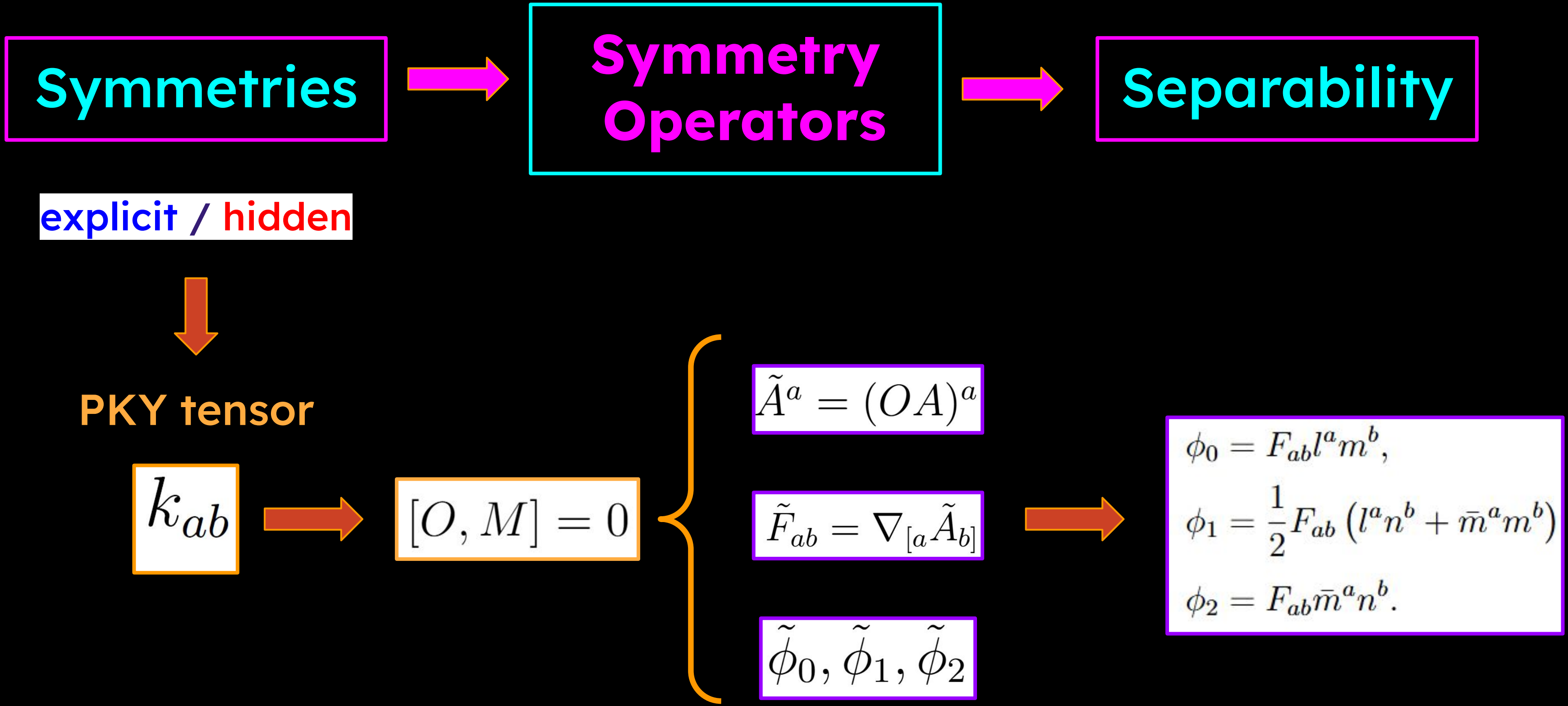
Linear combination **“plus”** :

$$\boxed{\tilde{A}^{+}_{a} = \tilde{A}^{(q_2)}_{a} + \tilde{A}^{(q_3)}_{a}}$$

$$\tilde{A}^{+}_{a} = (k,\, F)\,\xi_a - 6F_{[b}{}^{c}k_{a]c}\xi^b - \nabla_b\,(k\cdot k\cdot F)^{b}{}_{a}\,.$$

Its field strength then takes the form:

$$\begin{aligned}\tilde{F}^{+}_{da} = {} & 6F_{bd}\xi_a\xi^b - 6F_{ad}\xi_b\xi^b + 6F_{ab}\xi^b\xi_d + 3k_d{}^c\xi^b(\nabla_a F_{bc}) - 3k_b{}^c\xi^b(\nabla_a F_{dc}) \\ & - \xi_d\nabla_a(k,F) - 3F_{dc}k_b{}^c(\nabla_a\xi^b) + 3F_{bc}k_d{}^c(\nabla_a\xi^b) - (k,F)(\nabla_a\xi_d) \\ & - 3k_b{}^c\xi^b(\nabla_d F_{ac}) - 3k_a{}^c\xi^b(\nabla_d F_{bc}) + \xi_a\nabla_d(k,F) + (k,F)(\nabla_d\xi_a) \\ & - 3F_{bc}k_a{}^c(\nabla_d\xi^b) + 3F_{ac}k_b{}^c(\nabla_d\xi^b) - \nabla_d\nabla_b(k,k,F)^b{}_a + \nabla_a\nabla_b(k,k,F)^b{}_d.\end{aligned}$$



Symmetries → **Symmetry Operators** → **Separability**

explicit / hidden

PKY tensor

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$$\tilde{A}^a = (OA)^a$$

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$$\tilde{\phi}_0, \tilde{\phi}_1, \tilde{\phi}_2$$

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$$\phi_2 = F_{ab} \bar{m}^a n^b.$$

Maxwell scalars in GHP formalism

Projecting into the ***null tetrad***, using ***Maxwell*** equations, ***Teukolsky master*** equation and ***identities:***

$$\tilde{\phi}_0^- = \text{þ}\text{þ}\,\bar{\kappa}_1^2\bar{\phi}_2\,,$$

$$\tilde{\phi}_1^- = (\text{þ}\eth' + \tau'\text{þ})\,\bar{\kappa}_1^2\bar{\phi}_2\,,$$

$$\tilde{\phi}_2^- = \eth'\eth'\bar{\kappa}_1^2\bar{\phi}_2\,.$$

Known result from Debye potential theory

$$\tilde{\phi}_0^+ = [(\text{þ} - \bar{\rho})(\text{þ}' + 2\bar{\rho}' - \rho') + (\eth - \bar{\tau}')(\eth' + 2\bar{\tau} - \tau')]\,\kappa_1\bar{\kappa}_1\phi_0\,,$$

$$\tilde{\phi}_1^+ = (\text{þ}'\eth' + \bar{\tau}\text{þ}' + 2\bar{\rho}'\bar{\tau} - 2\rho'\tau')\,\kappa_1\bar{\kappa}_1\phi_0 + (\text{þ}\eth + \bar{\tau}'\text{þ} + 2\bar{\rho}\bar{\tau}' - 2\rho\tau)\,\kappa_1\bar{\kappa}_1\phi_2\,,$$

$$\tilde{\phi}_2^+ = [(\text{þ}' - \bar{\rho}')(\text{þ} + 2\bar{\rho} - \rho) + (\eth' - \bar{\tau})(\eth + 2\bar{\tau}' - \tau)]\,\kappa_1\bar{\kappa}_1\phi_2.$$

NEW!

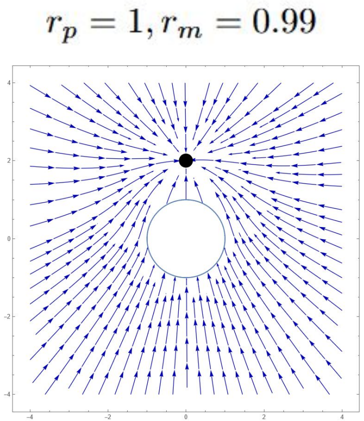
Action on fields

Eigenproblem summary: single-mode Maxwell field

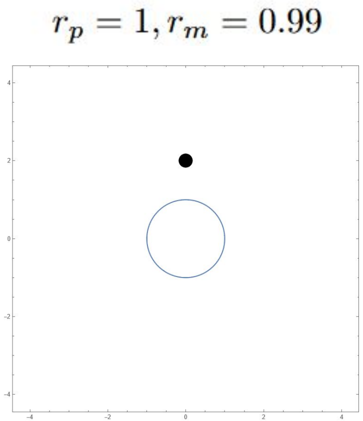
$$
\begin{aligned}
\mathcal{L}_\xi \boldsymbol{F}^{\omega}_{l,m} &= -i\omega\, \boldsymbol{F}^{\omega}_{l,m}\,, \\
\mathcal{L}_\eta \boldsymbol{F}^{\omega}_{l,m} &= im\, \boldsymbol{F}^{\omega}_{l,m}\,, \\
\mathcal{O}^+ \boldsymbol{F}^{\omega}_{l,m} &= \lambda^{-1,\omega}_{l,m}\, \boldsymbol{F}^{\omega}_{l,m}\,, \\
\mathcal{O}^-\, \boldsymbol{F}^{\omega}_{l,m} &= -\mathcal{C}^{-1,-\omega}_{l,-m}\, \boldsymbol{F}^{-\omega}_{l,-m}\,, \\
\mathcal{O}^-\mathcal{O}^-\, \boldsymbol{F}^{\omega}_{l,m} &= \mathcal{C}^{-1,-\omega}_{l,-m}\mathcal{C}^{1,\omega}_{l,m}\, \boldsymbol{F}^{\omega}_{l,m} = \left(\mathcal{C}^{-1,\omega}_{l,m}\right)^2 \boldsymbol{F}^{\omega}_{l,m}\,.
\end{aligned}
$$

$$
\mathcal{C}^{\pm1,\omega}_{l,m} = \sqrt{\left(\lambda^{\pm1,\omega}_{l,m}\right)^2 + 4ma\omega - 4a^2\omega^2}
$$

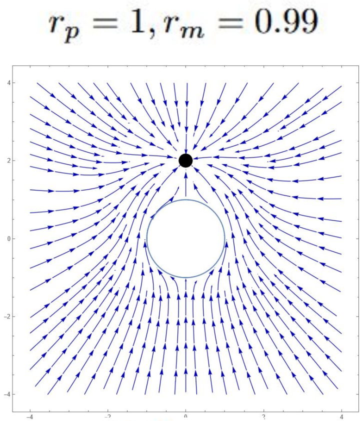
Test point particle field: ZAMO observers


$r_p = 1, r_m = 0.99$

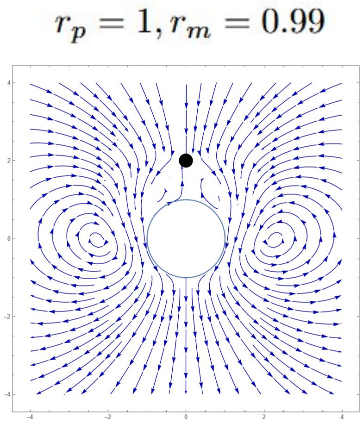
(a) $\boldsymbol{E}_{ZAMO}$


$r_p = 1, r_m = 0.99$

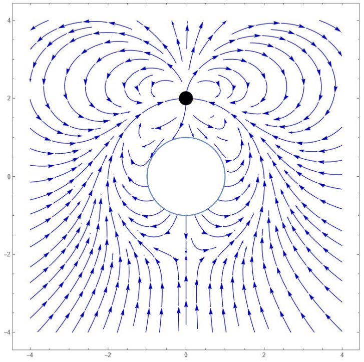
(b) $\boldsymbol{B}_{ZAMO}$


$r_p = 1, r_m = 0.99$

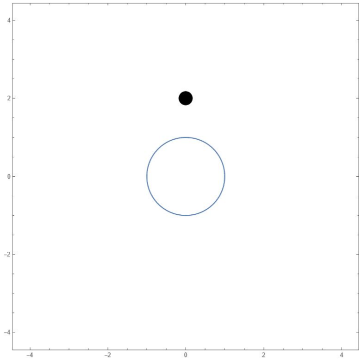
(c) $\boldsymbol{E}_{ZAMO}$


$r_p = 1, r_m = 0.99$

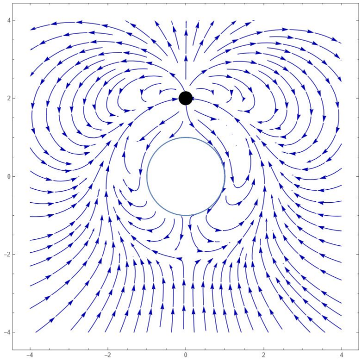
(d) $\boldsymbol{B}_{ZAMO}$



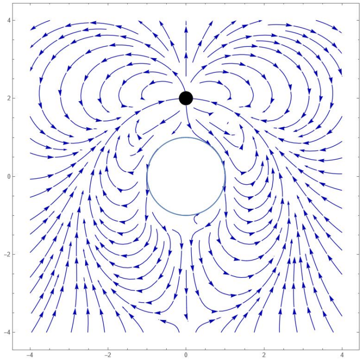
(e) $O^{+}\boldsymbol{E}_{ZAMO}$



(f) $O^{+}\boldsymbol{B}_{ZAMO}$



(g) $O^{+}\boldsymbol{E}_{ZAMO}$



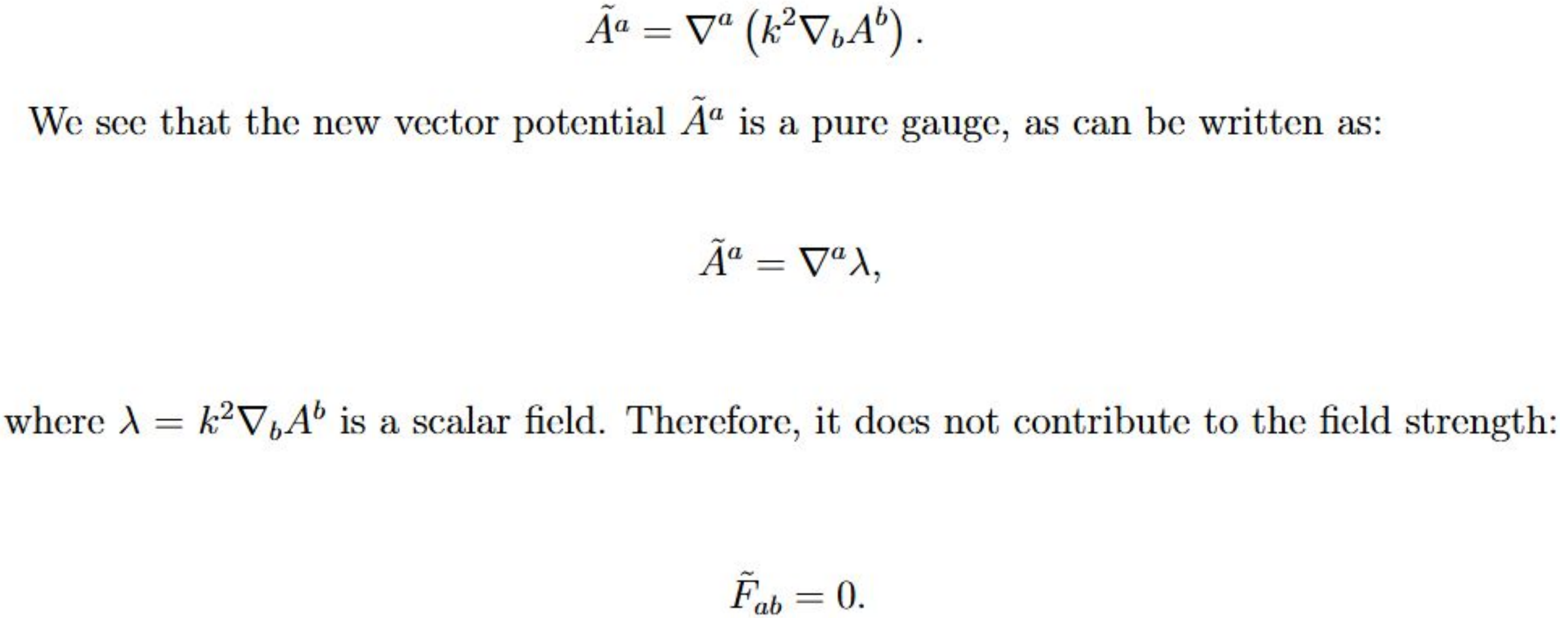
(h) $O^{+}\boldsymbol{B}_{ZAMO}$

Summary:

- We study a class of symmetry operators acting on vector perturbations in the Kerr.
- We started from operators constructed from the PKY tensor.
- We translated the operators into NP/GHP formalism.
- We found a new “plus” operator.
- When applied to a single-mode solution, the new operators naturally reproduce the Teukolsky separation constants.
- Operators are useful as projectors on single-modes (isolate specific modes).
- This eigenstructure ties our abstract symmetry operators back to the familiar Teukolsky separation constants.

Thank you!
Questions?

Quadratic operator 1



$$\tilde{A}^a = \nabla^a \left(k^2 \nabla_b A^b \right).$$

We see that the new vector potential $\tilde{A}^a$ is a pure gauge, as can be written as:

$$\tilde{A}^a = \nabla^a \lambda,$$

where $\lambda = k^2 \nabla_b A^b$ is a scalar field. Therefore, it does not contribute to the field strength:

$$\tilde{F}_{ab} = 0.$$

Quadratic in *k*:

Quadratic operator 2

$$\tilde{A}^a = (O^{(v)}_{q2})^a{}_b A^b.$$

$$\tilde{A}^{(q2)}_a = 2\,(k,\,F)\,\xi_a - 4\,F_{bc}k_a{}^c\,\xi^b + 6\,F_{ac}k_b{}^c\,\xi^b - 2\nabla_b\,(k\cdot k\cdot F)^b{}_a - k_{ab}\nabla^b\,(k,\,F)$$

Quadratic operator 3

$$\tilde{A}^a = (O^{(v)}_{q3})^a{}_b A^b.$$

$$\tilde{A}^{(q3)}_a = k_a{}^b\Big(\nabla_b\,(k,F) + 2F_{bc}\xi^c\Big).$$

How the new operator act on a single mode?

$$\tilde{\phi}_0^+ = \left[(\text{þ} - \bar{\rho})\,(\text{þ}' + 2\bar{\rho}' - \rho') + (\eth - \bar{\tau}')\,(\eth' + 2\bar{\tau} - \tau')\right] \kappa_1 \bar{\kappa}_1 \phi_0\,,$$

$$\tilde{\phi}_1^+ = (\text{þ}'\eth' + \bar{\tau}\text{þ}' + 2\bar{\rho}'\bar{\tau} - 2\rho'\tau')\,\kappa_1\bar{\kappa}_1\phi_0 + (\text{þ}\eth + \bar{\tau}'\text{þ} + 2\bar{\rho}\bar{\tau}' - 2\rho\tau)\,\kappa_1\bar{\kappa}_1\phi_2\,,$$

$$\tilde{\phi}_2^+ = \left[(\text{þ}' - \bar{\rho}')\,(\text{þ} + 2\bar{\rho} - \rho) + (\eth' - \bar{\tau})\,(\eth + 2\bar{\tau}' - \tau)\right] \kappa_1\bar{\kappa}_1\phi_2.$$

Let us have a solution – a single mode $\boldsymbol{F}^{\omega}_{l,m}$ with projections:

$$\phi_0 = \frac{1}{2}\, e^{-i\omega t}\, R^{1,\omega}_{l,m}(r)\, Y^{1,\omega}_{l,m}(\theta,\phi)\,,$$

$$\phi_1 = \frac{e^{-i\omega t}\sin\theta}{8\rho^2 MK}\Bigg[2a\,(m - \omega(a + ir\cos\theta))\,\left(P^{-1,\omega}_{l,m}Y^{-1,\omega}_{l,m} + P^{1,\omega}_{l,m}Y^{1,\omega}_{l,m}\right)$$

$$- i\rho\,\lambda^{-1,\omega}_{l,m}\left(P^{-1,\omega}_{l,m}Y^{-1,\omega}_{l,m} - P^{1,\omega}_{l,m}Y^{1,\omega}_{l,m}\right) - i\rho\,\mathcal{C}^{-1,\omega}_{l,m}\left(-P^{1,\omega}_{l,m}Y^{-1,\omega}_{l,m} + P^{-1,\omega}_{l,m}Y^{1,\omega}_{l,m}\right)\Bigg]\,,$$

$$\phi_2 = \frac{1}{2}\frac{1}{\rho^2}\, e^{-i\omega t}\, R^{-1,\omega}_{l,m}(r)\, Y^{-1,\omega}_{l,m}(\theta,\phi).$$

How the new operator act on a single mode?

$$
\begin{aligned}
\tilde{\phi}_0^+ &= \left[(\text{þ} - \bar{\rho})\,(\text{þ}' + 2\bar{\rho}' - \rho') + (\eth - \bar{\tau}')\,(\eth' + 2\bar{\tau} - \tau')\right] \kappa_1 \bar{\kappa}_1 \phi_0\,, \\
\tilde{\phi}_1^+ &= (\text{þ}'\eth' + \bar{\tau}\text{þ}' + 2\bar{\rho}'\bar{\tau} - 2\rho'\tau')\,\kappa_1 \bar{\kappa}_1 \phi_0 + (\text{þ}\eth + \bar{\tau}'\text{þ} + 2\bar{\rho}\bar{\tau}' - 2\rho\tau)\,\kappa_1 \bar{\kappa}_1 \phi_2\,, \\
\tilde{\phi}_2^+ &= \left[(\text{þ}' - \bar{\rho}')\,(\text{þ} + 2\bar{\rho} - \rho) + (\eth' - \bar{\tau})\,(\eth + 2\bar{\tau}' - \tau)\right] \kappa_1 \bar{\kappa}_1 \phi_2.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\phi}_0^+ &= \frac{1}{2}\, e^{-i\omega t}\, \lambda_{l,m}^{-1,\omega}\, R_{l,m}^{1,\omega}(r)\, Y_{l,m}^{1,\omega}(\theta,\phi) \\
\tilde{\phi}_1^+ &= \frac{e^{-i\omega t}\sin\theta}{8\rho^2 M K}\, \lambda_{l,m}^{-1,\omega} \Big[2a\,(m - \omega(a + ir\cos\theta))\,\big(P_{l,m}^{-1,\omega} Y_{l,m}^{-1,\omega} + P_{l,m}^{1,\omega} Y_{l,m}^{1,\omega}\big) \\
&\quad - i\rho\,\lambda_{l,m}^{-1,\omega}\,\big(P_{l,m}^{-1,\omega} Y_{l,m}^{-1,\omega} - P_{l,m}^{1,\omega} Y_{l,m}^{1,\omega}\big) - i\rho\,\mathcal{C}_{l,m}^{-1,\omega}\,\big(-P_{l,m}^{1,\omega} Y_{l,m}^{-1,\omega} + P_{l,m}^{-1,\omega} Y_{l,m}^{1,\omega}\big) \Big] \\
\tilde{\phi}_2^+ &= \frac{1}{2}\frac{1}{\rho^2}\, e^{-i\omega t}\, \lambda_{l,m}^{-1,\omega}\, R_{l,m}^{-1,\omega}(r)\, Y_{l,m}^{-1,\omega}(\theta,\phi),
\end{aligned}
$$

where we have quite a trivial action of the operator $\mathcal{O}^+$ on a single mode as follows:

$$
\boldsymbol{F}^+ = \lambda_{l,m}^{-1,\omega}\, \boldsymbol{F}_{l,m}^{\omega}.
$$

Parity operator

Let us consider an active coordinate transformation $\boldsymbol{\mathcal{P}}$ of the field $\boldsymbol{F}^{\omega}_{l,m}$ given by:

$$\theta \to \pi - \theta\,, \qquad\qquad \phi \to \pi + \phi,$$

together with the Jacobian diag$(1, 1, -1, 1)$ which yields a new field $\tilde{\boldsymbol{F}}^{\omega}_{l,m} = \boldsymbol{\mathcal{P}}\,\boldsymbol{F}^{\omega}_{l,m}$. One finds:

$$\boldsymbol{\mathcal{O}}^{-}\,\boldsymbol{F}^{\omega}_{l,m} = -(-1)^{l+m}\,\mathcal{C}^{-1,-\omega}_{l,-m}\,\tilde{\boldsymbol{F}}^{\omega}_{l,m}.$$

From where we read that $\boldsymbol{\mathcal{O}}^{-} \sim -(-1)^{l+m}\boldsymbol{\mathcal{P}}$ is proportional to the parity operator.