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Axisymmetric spacetimes circularity and beyond

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Motivation: Going beyond Kerr

- In General Relativity the Kerr solution describes vacuum rotating black hole
- It is important to construct deformations of the Kerr spacetime to have a **benchmark for testing General Relativity** and look for **signatures of modified gravity**.
- One approach is to find (construct) rotating BH solutions
- Alternative approach is to be agnostic and to construct an *ad hoc* metric that describes a deviation from the Kerr metric

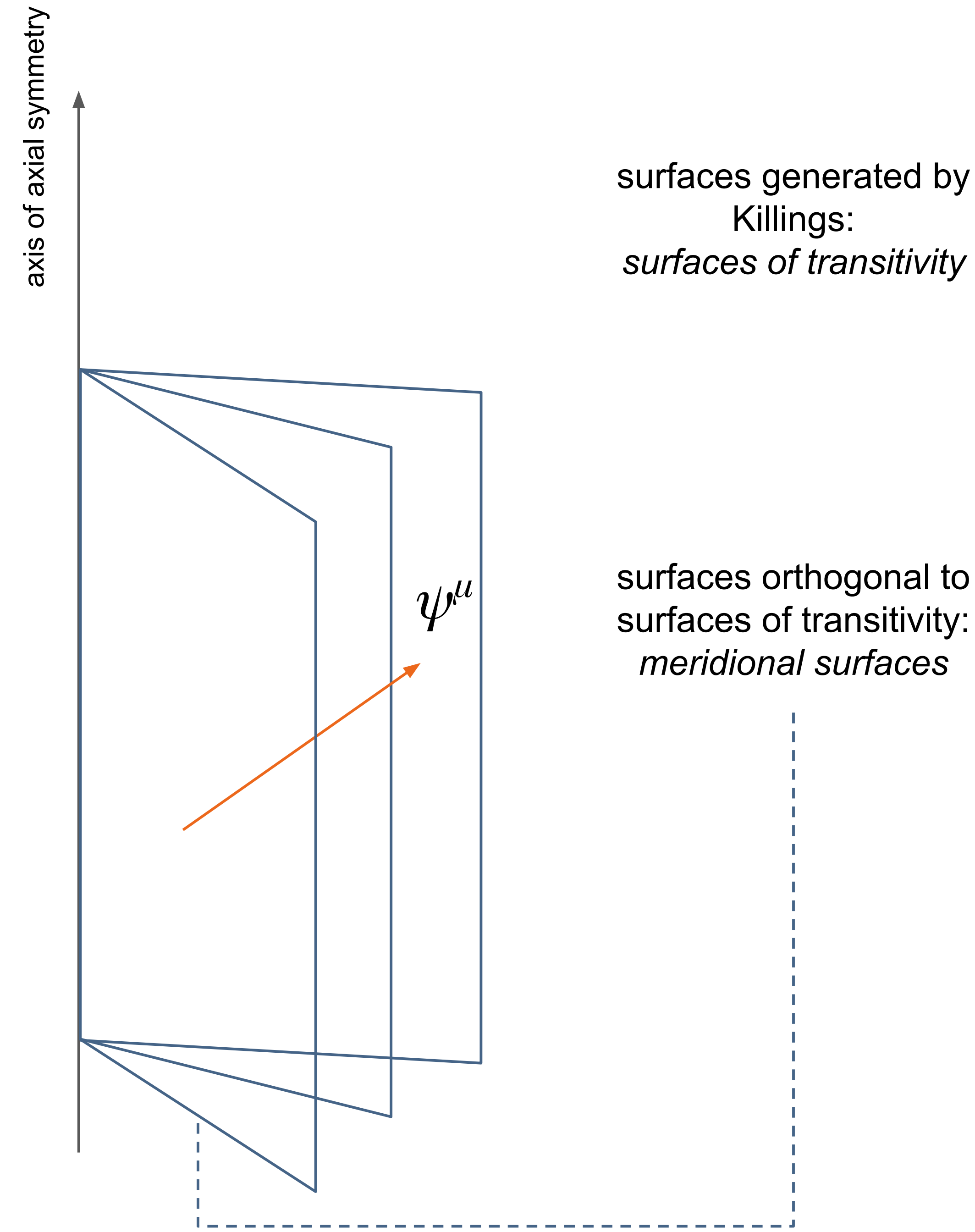
Motivation: Circularity

- Circularity is accidental symmetry of Kerr solution. A symmetry that is in addition to stationarity and axisymmetry `[Also the talk by Cynthia Belen Arias Pruna]`
- In GR, for vacuum solutions circularity is automatic `[Even asymptotically "swirling" solution, talk by José Barrientos]`
- Beyond GR : do we keep circularity or not? Usually yes, but this is not always justified.

Circularity vs. non-circularity

- Most general stationary axisymmetric spacetime: two Killing vectors associated to the symmetries
  - ξ^μ (stationarity)
  - ψ^μ (axial symmetry)

- A spacetime is said to be circular if it can be foliated by codimension-2 surfaces orthogonal to the Killing vectors.



by Frobenius' theorem:

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$
$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

N.B. another definition of circularity involves matter. They are "equivalent".

Carter, Black holes equilibrium states, Les Houces lecture notes'73

Circularity vs. non-circularity

If such foliation exist, one can choose coordinates adapted to foliation

The metric is block diagonal

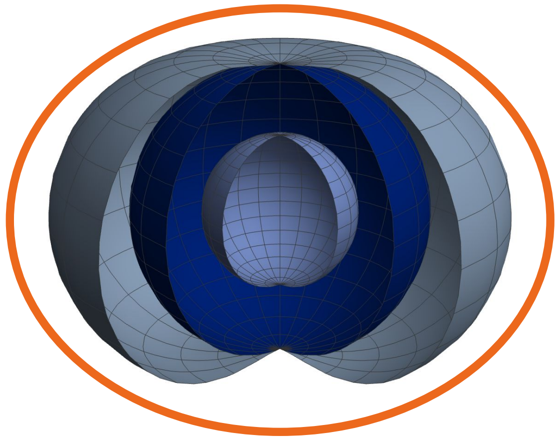
$$g_{\mu\nu} = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

$t - \phi$ symmetry

4 functions of two variables

For Kerr metric

Boyer-Lindquist coordinates



Why circularity?

- Vacuum GR solutions are circular

- Some solutions to alternative theories of gravity are circular:
  – stealth solutions, whereby the metric is that of Kerr;
  – numerical solutions in dilatonic EGB theory;
  – perturbative rotating solutions in sEGB and dynamical Chern-Simons gravity;

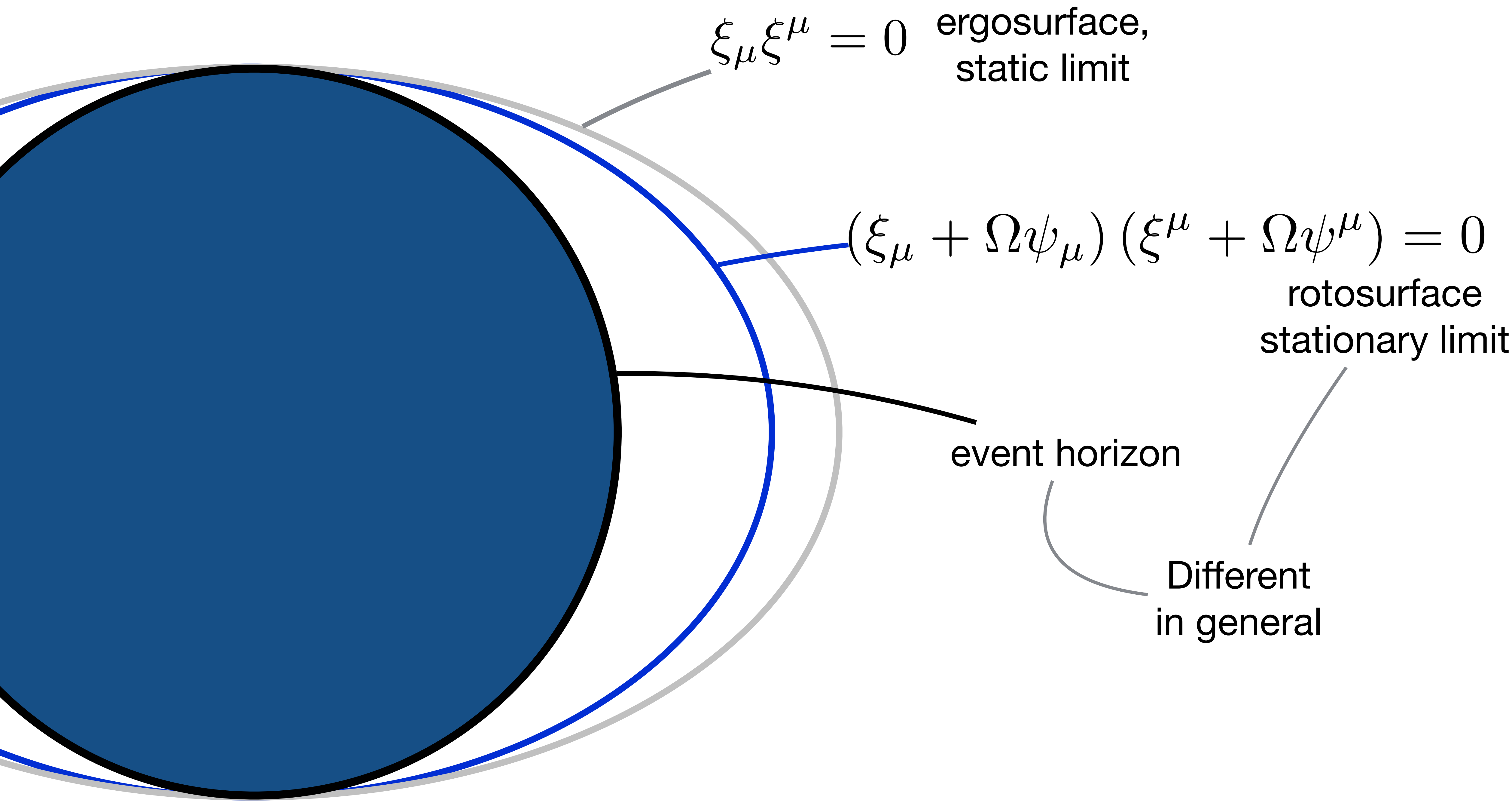
- beyond-GR solutions that are continuously connected to GR are circular [Y. Xie et al'21]

Most importantly — Simplicity

- Metric is block diagonalizable

- And other features...

Notable surfaces



Circularity means simplicity

If circularity, then:

- rotosurface $\equiv$ horizon
- frame dragging $\Omega =$ const. on the horizon (Rigidity theorem)
- event horizon is Killing
- different notions of surface gravity agree
- surface gravity is constant (0th law of BH mechanics)

In addition:

Circularity is necessary (although not sufficient) condition for separability

Circular metrics are useful for phenomenological studies

Why NONcircularity?

- Even in GR non-vacuum solutions may have a non-circular metric `Gourgoulhon, Bonazzola'93`

- Examples of non-circular solutions:
  – Disformed Kerr black hole `Anson, EB, Charmousis, Hassaine'21`;
  – Numerical rotating BHs in Einstein-Aether `Adam, Figueras, Jacobson, Wiseman'21`

- Circularity is an additional assumption. In modified gravity this assumption translates to the symmetry assumption on extra fields.
  – E.g. in scalar-tensor theory one needs $\xi^\mu \partial_\mu \varphi = \psi^\mu \partial_\mu \varphi = 0$ to fall into class of theories with circular solutions.

Assuming circularity *a priori* is often not justified

What happens when circularity is broken?

Find circularity condition?

The circularity condition are PDEs

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$
$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

Given a metric it is easy to check circularity

It is difficult to construct simple non-circular metrics

! Need to "solve" conditions to get algebraic relations?

⇦ Depends on choice of "good" gauge

Gauge choice for general axisymmetric stationary metric

- Coordinates adapted to symmetries, i.e. v and ϕ along the Killing orbits,

$$\xi^\mu \partial_\mu = \partial_v \quad \text{and} \quad \psi^\mu \partial_\mu = \partial_\phi$$

Other coordinates r and θ, so that

$$g_{\mu\nu}\left(x^\alpha\right) = g_{\mu\nu}(r,\theta)\,.$$

- These coordinates are defined up to

$$\tilde{v} = v + V\left(r,\theta\right)$$
$$\tilde{r} = R\left(r,\theta\right)$$
$$\tilde{\theta} = \Theta\left(r,\theta\right)$$
$$\tilde{\phi} = \phi + \Phi\left(r,\theta\right)$$

$$g_{\mu\nu} = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

where $* = *(r,\theta)$

Most general choice of coords adapted to the symmetries (10 components of metric)

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In [Delaporte, Eichhorn & Held'22] this form was suggested as an *ansatz*

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Gauge choice: kill 4 components of the metric?

It is possible locally (using *Cauchy-Kovalevskaya theorem)*

[EB & Mazza'25]

Solving circularity conditions

Kerr-like gauge is great

$$\xi_{[\mu}\psi_\nu\partial_\rho\xi_{\sigma]} = 0$$
$$\xi_{[\mu}\psi_\nu\partial_\rho\psi_{\sigma]} = 0$$

Plug into Kerr (orthogonal) gauge

Long calculations

$$g^{vr} = f(r)g^{rr}$$
$$g^{r\phi} = h(r)g^{rr}$$

[EB & Mazza'25]

N.B. The use of the upper indices !

Why is this interesting?

$$g^{vr} = f(r) g^{rr}$$

$$g^{r\phi} = h(r) g^{rr}$$

Solved circularity conditions

Soft breaking of non-circularity

Toy models to study effects of circularity breaking

Examples (see in arxiv:2505.08880) of minimal and not-so-minimal deformations of Kerr metric

Conclusions and outlook

- Gauge choice: orthogonal and Kerr-like gauges can be chosen
- Conditions of circularity are solved
- Examples

- Non-constant surface gravity: Hawking radiation and thermodynamics?
- Observational consequences: QNM, geodesics, shadows
- Superradiance
- Residual gauge: can we simplify further?