# Kerr–Levi-Civita black hole: A new rotating spacetime in vacuum

Based on 2506.07166 **[gr-qc]**, **JB**, A. Cisterna, M. Hassaine, K. Müller and K. Pallikaris

José Barrientos

Universidad de Tarapacá
Institute of Mathematics, Czech Academy of Sciences

September 2, 2025, NEB–21, Corfu

UNIVERSIDAD DE TARAPACÁ
*Universidad del Estado*
Sede Iquique

Agencia Nacional de Investigación y Desarrollo
Ministerio de Ciencia, Tecnología, Conocimiento e Innovación
Gobierno de Chile

# Introduction

- To date, only two exact solutions with well-defined static limits are known to describe the exterior gravitational field of a spinning mass in vacuum.
- The first is the Kerr metric,[1] which remains the most astrophysically relevant solution to Einstein's field equations, underpinning much of our theoretical understanding of rotating black holes.
- The second is its less familiar generalization, the Tomimatsu–Sato metric.[2]
- Unlike the Kerr metric, the Tomimatsu–Sato solution does not reduce to the Schwarzschild geometry in the limit of vanishing angular momentum, but to a class of Weyl metrics known as Zipoy–Voorhees spacetimes.[3]
- The latter represents in the Weyl representation, a finite thin rod of mass $M$ and size $2l$, with an arbitrary linear density.

[1] R.P. Kerr, PRL (1963)

[2] A. Tomimatsu and H. Sato, PRL (1972)

[3] D.M. Zipoy, JMP (1966). B.H. Voorhees, PRD (1970)

- Stationary and axisymmetric geometries in four dimensions are characterized by the action of a group $\mathbb{R} \times \mathbf{SO(2)}$ under which the spacetime metric remains invariant.
- If the orbits of $\mathbb{R} \times \mathbf{SO(2)}$ (surfaces of transitivity) are everywhere orthogonal to a family of hypersurfaces defined by the remaining, non-Killing coordinates of the spacetime (meridional surfaces), then the spacetime is said to be circular, and the action of the group is said to be orthogonally transitive.
- Upon establishing circularity, stationarity, and axisymmetry, the most general spacetime is described by the LWP metric, either in its standard electric form or in its magnetic representation.

- In the canonical Weyl coordinates denoted by $\{t, \rho, z, \phi\}$, they read

$$\mathrm{d}s^2 = -f\left(\mathrm{d}t - \omega\,\mathrm{d}\phi\right)^2 + \frac{1}{f}\left[\mathrm{e}^{2\gamma}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2\,\mathrm{d}\phi^2\right], \tag{1}$$

$$\mathrm{d}s^2 = f\left(\mathrm{d}\phi - \omega\,\mathrm{d}t\right)^2 + \frac{1}{f}\left[\mathrm{e}^{2\gamma}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) - \rho^2\,\mathrm{d}t^2\right], \tag{2}$$

  where $f$, $\omega$, and $\gamma$ are functions of $\rho$ and $z$ only.
- It can be shown that Einstein's equations reduce to the complex Ernst equation[4]

$$\mathrm{Re}\left(\mathcal{E}\right)\nabla^2\mathcal{E} = \boldsymbol{\nabla}\mathcal{E}\cdot\boldsymbol{\nabla}\mathcal{E}, \tag{3}$$

  where $\mathcal{E}$ is the complex Ernst gravitational potential

$$\mathcal{E} = -f - i\chi \tag{4}$$

  and $\chi$ is provided by the relevant twist equation

$$\hat{\boldsymbol{\phi}} \times \boldsymbol{\nabla}\chi = -\frac{f^2}{\rho}\boldsymbol{\nabla}\omega. \tag{5}$$

[4] F.J. Ernst, PR (1968)

The formalism offers two main advantages:

1. We reduce the problem to solving a PDE system in Euclidean space with cylindrical coordinates.
2. Einstein(–Maxwell) equations can be shown to display a set of hidden Lie point symmetries. These symmetries are[5]

$$\mathrm{G}_1[a] : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) \quad := \quad (\mathcal{E} + ia, \Phi_0), \tag{6a}$$

$$\mathrm{G}_2[\alpha] : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) \quad := \quad (\mathcal{E}_0 - 2\overline{\alpha}\Phi_0 - \alpha\overline{\alpha}, \Phi_0 + \alpha), \tag{6b}$$

$$\mathrm{D}[\epsilon] : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) \quad := \quad (\epsilon\bar{\epsilon}\mathcal{E}_0, \epsilon\Phi_0), \tag{6c}$$

$$\mathrm{E}[c] : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) \quad := \quad \frac{(\mathcal{E}_0, \Phi_0)}{1 + ic\mathcal{E}_0}, \tag{6d}$$

$$\mathrm{H}[\beta] : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) \quad := \quad \frac{(\mathcal{E}_0, \Phi_0 + \beta\mathcal{E}_0)}{1 - 2\overline{\beta}\Phi_0 - \beta\overline{\beta}\mathcal{E}_0}. \tag{6e}$$

From known solutions $(\mathcal{E}_0, \Phi_0)$ we can construct new solutions $(\mathcal{E}, \Phi)$ with $\mathrm{E}[c]$, $\mathrm{H}[\beta]$.

[5] Exact solutions of Einstein's field equations, Cambridge Monographs on Mathematical Physics (2003)

- When applying Ehlers or Harrison on a (asymptotically flat) seed spacetime written in electric LWP form the resulting spacetime is asymptotically flat:
  - i. Ehlers: Schwarzschild $\underset{\mathrm{E}[c]}{\to}$ Taub–NUT
  - ii. Harrison: Schwarzschild $\underset{\mathrm{H}[\beta]}{\to}$ Reissner–Nordström
- When applying on a seed in a magnetic LWP form the asymptotic behaviour of the seed changes to a Levi-Civita-like spacetime

$$\mathrm{d}s^2 = -\rho^{4\sigma}\,\mathrm{d}t^2 + k^2\rho^{4\sigma(2\sigma-1)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^{2(1-2\sigma)}\,\mathrm{d}\phi^2. \quad (7)$$

  - i. Ehlers: Minkowski $\underset{\mathrm{E}[c]}{\to}$ Swirling background

$$\mathrm{d}s^2 = \frac{\rho^2}{1+j^2\rho^4}\left(\mathrm{d}\phi + 4jz\,\mathrm{d}t\right)^2 + \left(1+j^2\rho^4\right)\left(-\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2\right).$$

  - ii. Harrison: Minkowski $\underset{\mathrm{H}[\beta]}{\to}$ Melvin background

$$\mathrm{d}s^2 = \left[1 + \frac{E^2}{4}\rho^2\right]^2\left(-\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \left[1 + \frac{E^2}{4}\rho^2\right]^{-2}\rho^2\,\mathrm{d}\phi^2, \quad A = Ez\,\mathrm{d}t.$$

- Relevant to our construction is the discrete inversion transformation that arises from a Weyl rescaling, a gravitational gauge transformation, and an Ehlers in the limit where the Ehlers parameter is taken to infinity[6]

$$\mathrm{I} : \mathcal{E}_0 \mapsto \mathcal{E} := \frac{1}{\mathcal{E}_0}. \tag{8}$$

- For a seed Ernst potential $\mathcal{E}_0 = -f_0 - i\chi_0$, the inversion acts as:

$$f = \frac{f_0}{f_0^2 + \chi_0^2}, \qquad \chi = -\frac{\chi_0}{f_0^2 + \chi_0^2}. \tag{9}$$

- Buchdahl gave an intuitive metric version in the static case:[7] If

$$\mathrm{d}s_0^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = F(x^k)(\mathrm{d}x^a)^2 + g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{10}$$

is a vacuum solution, then the reciprocal

$$\mathrm{d}s^2 = F^{-1}(\mathrm{d}x^a)^2 + F^2 g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{11}$$

is also a solution.

[6] J. Ehlers, Colloq. Int. CNRS (1962)

[7] H.A. Buchdahl, AJP (1956)

# The Kerr–Levi-Civita spacetime

- Similar to Ehlers and Harrison: It can be proven that the electric inversion does not transform any solution of the Plebański–Demiański family. The Ernst potential remains asymptotically flat. This is consistent with uniqueness theorems.[8]
- Then the importance of expressing the seed metric in a magnetic form becomes fundamental.
- Example: Schwarzschild–Levi-Civita black hole with Buchdahl theorem.[9]
- We construct the Kerr–Levi-Civita spacetime starting from the Kerr metric as a seed, expressed in magnetic LWP form with spherical-like coordinates $\{t, r, x = \cos\theta, \phi\}$.

---

[8] B. Carter, PRL (1971)

[9] **J.B.**, A. Cisterna, M. Hassaine, and J. Oliva, EPJC (2024)

Explicitly:

$$\mathrm{d}s_0^2 = f_0(\mathrm{d}\phi - \omega_0\,\mathrm{d}t)^2 - \frac{\Delta_r\Delta_x\,\mathrm{d}t^2}{f_0} + \frac{\mathrm{e}^{2\gamma_0}}{f_0}\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}x^2}{\Delta_x}\right), \qquad (12)$$

where

$$\begin{aligned} f_0(r,x) &= \Delta_x\frac{(r^2+a^2)^2 - a^2\Delta_r\Delta_x}{\varrho^2}, \\ \omega_0(r,x) &= a\frac{(r^2+a^2) - \Delta_r}{(r^2+a^2)^2 - a^2\Delta_r\Delta_x}, \\ \mathrm{e}^{2\gamma_0(r,x)} &= \Delta_x\left[(r^2+a^2)^2 - a^2\Delta_r\Delta_x\right], \\ \Delta_r(r) &= r^2 - 2mr + a^2, \quad \Delta_x(x) = 1 - x^2, \end{aligned} \qquad (13)$$

and $\varrho^2(r,x) = r^2 + a^2x^2$.

The function $f$ follows directly, while $\omega$ is obtained from the seed $\omega_0$ via the twist equation (5), giving $\chi_0$.

Thus, we end up with

$$f(r,x) = \Delta_x \varrho^2 \frac{(r^2+a^2)^2 - a^2\Delta_r\Delta_x}{4m^2a^2x^2\left[a^2\Delta_x^2 + \varrho^2(\Delta_x+2)\right]^2 + \Delta_x^2\left[(r^2+a^2)^2 - a^2\Delta_r\Delta_x\right]^2},$$

$$\omega(r,x) = \frac{(2a^2m - 3a^2r + r^3)\Delta_r x^4 - 6r(a^2+r^2)\Delta_r x^2 + (2a^2m + a^2r + r^3)(a^2 - 6mr - 3r^2)}{\Delta_r a^2x^2 + r(2a^2m + a^2r + r^3)} \times (-2ma).$$

The Kerr–Levi-Civita spacetime in spherical-like coordinates is given by

$$\mathrm{d}s^2 = f(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 - \frac{\Delta_r\Delta_x\,\mathrm{d}t^2}{f} + \frac{\mathrm{e}^{2\gamma_0}}{f}\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}x^2}{\Delta_x}\right).$$

- Since $g_{\phi\phi} \equiv f > 0$, the azimuthal Killing vector $\partial_\phi$ is spacelike for $r > 0$, becoming null only on the axis $x = \pm 1$. Hence, no closed timelike curves arise.
- Introducing a new coordinate system $\{t, \tilde{\rho}, \tilde{z}, \phi\}$, the induced metric on the slices of constant $t$ and $\tilde{z}$, near the symmetry axis, becomes

$$\mathrm{d}s^2 \sim \mathrm{d}\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{256a^4m^4}\,\mathrm{d}\phi^2. \tag{14}$$

- Since $|\boldsymbol{\partial}_\phi|^2 \sim \tilde{\rho}^2$, there is no Misner string.
- However, there is a defect angle $2\pi(1-\vartheta)$ where $\vartheta = \pm 1/(16a^2m^2)$. To deal with that, we can redefine

$$\phi = 16a^2m^2\varphi, \tag{15}$$

and take $\varphi$ as our new azimuthal coordinate. Then, the new spacetime is free of conical singularities as well.

- In coordinates $\{t, \rho, z, \varphi\}$, the asymptotic form of the metric reads

$$\mathrm{d}s^2 \underset{\rho,z\to\infty}{\sim} \mathrm{d}s^2_{\mathrm{LC}} - 64a^3m^3 \frac{3\rho^4 + 12\rho^2 z^2 + 8z^4}{\rho^2(\rho^2 + z^2)^{3/2}} \,\mathrm{d}t\,\mathrm{d}\varphi, \tag{16}$$

where

$$\mathrm{d}s^2_{\mathrm{LC}} = \rho^4 \left(-\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \frac{256a^4m^4}{\rho^2}\,\mathrm{d}\varphi^2. \tag{17}$$

- The latter belongs to Levi-Civita spacetimes.
- Since the asymptotic geometry is a rotating generalization of the LC geometry, we may formally address the solution as the Kerr–LC spacetime.

- Two limits can be taken in the Kerr–Levi-Civita spacetime:
  1. For $m = 0$, the solution is nothing but the LC background (for $\sigma = 1 = k$)

$$\mathrm{d}s^2 = r^4 \sin^4\theta \left[\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\right] + \frac{\mathrm{d}\phi^2}{r^2 \sin^2\theta}. \tag{18}$$

  2. For $a = 0$, we recover the Schwarzschild–Levi-Civita solution: a static black hole embedded in a LC cylindrical background

$$\mathrm{d}s^2 = r^4 \sin^4\theta \left[\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2\,\mathrm{d}\theta^2\right] + \frac{\mathrm{d}\phi^2}{r^2 \sin^2\theta}. \tag{19}$$

- In both cases, we end up with static spacetimes.
- Thus, a spinning mass placed in a Levi-Civita background drags it asymptotically, producing a rotation effect, similar to the one of swirling spacetimes. But with the crucial difference that the background of the Kerr–LC is non-rotating.

- The metric component $g_{rr}$ features poles as in Kerr black hole,

$$r_+ = m + \sqrt{m^2 - a^2}, \quad r_- = m - \sqrt{m^2 - a^2}, \tag{20}$$

- Curvature invariants are regular there! In fact, they are regular everywhere! (Schwarzschild–LC is singular all along the symmetry axis) There is no ring singularity like in Kerr.
- Due to the swirling-like asymptotic behavior of the spacetime, it is impossible to define a Killing vector that remains timelike everywhere. This is a phenomenon already observed in magnetized or swirling black hole spacetimes,[10] where the asymptotic dragging of inertial frames prevents a global timelike direction. Equivalently, the ergoregions extend to infinity in the Kerr–LC spacetime:

[10] G.W. Gibbons, A.H. Mujtaba, and C.N. Pope, CQG (2013). **J.B.** *et al*, EPJC (2024). A. Vigano, 2211.00436 (2022)

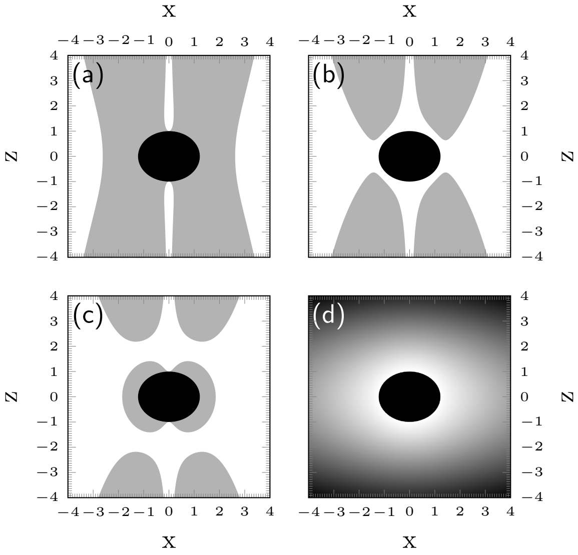


**Figure:** Cross section, taken at $\mathrm{y} = 0$, of the ergoregion (gray fill) dressing the event-horizon of a Kerr–LC black hole with $r_+ = 1$ and $r_- = 1/2$ in rotating frames with angular velocity $\alpha = 0$ (a), $\alpha = 1.64$ (b), and $\alpha = 3$ (c).

One last comment regarding the similarities with swirling spacetimes:

- In a Kerr spacetime, the asymptotic geometry is static; the angular velocity falls off like $\sim 1/r^3$. In the Kerr–LC spacetime, we have that

$$\Omega \underset{r\to\infty}{\sim} \frac{3+6x^2-x^4}{8am} r. \tag{21}$$

- It is rather reminiscent of what happens in spacetimes describing black holes embedded into a swirling universe.

# Final remarks

- We have constructed a novel rotating vacuum solution of Einstein's field equations: Kerr–Levi-Civita black hole.
- It is a rotating generalization of the recently studied Schwarzschild–Levi-Civita black hole.
- Completely regular: Free of curvature singularities, conical defects, spinning strings, and closed timelike curves.
- Similarities with swirling spacetimes; however, the frame-dragging in swirling black holes exists even if we remove the mass source, be it static or non-static, due to the intrinsic rotation of the background. On the other hand, dragging in the Kerr–LC spacetime is solely due to the angular momentum $J \equiv am$ of the Kerr seed.

Thank you!