A Directive for obtaining (Algebraically) General Solutions in General Relativity

based on the canonical Killing tensor forms

(Kokkinos & Papakostas, arXiv:2504.00202)

Dionysios Kokkinos

Department of Mechanical Engineering

Hellenic Mediterranean University

21st Conference Recent Developments in Gravity (NEB-21)

1 - 4 September 2025

Hellenic Society for Relativity, Gravitation and Cosmology (HSRGC)

Outline

- Introduction
- Canonical Forms of Killing Tensor
- Null tetrad transformations
- Type D solution
- Type I solution
- Summary & Discussion

Introduction

Any gravitational field theory is formulated in an action and

$$S = \frac{1}{2\kappa}\int \sqrt{-g}\ \left[R - 2\Lambda\right]\ dx^4 + \mathcal{S}_M \qquad \text{(GR)}$$

most of them reduces to General Relativity, the gravitational *Paradigm* of our time.

Introduction

Any gravitational field theory is formulated in an action and

$$\boldsymbol{S} = \frac{1}{2\boldsymbol{\kappa}} \int \sqrt{-\boldsymbol{g}} \; [\boldsymbol{R} - 2\boldsymbol{\Lambda}] \; \boldsymbol{dx}^4 + \boldsymbol{\mathcal{S}_M} \qquad \text{(GR)}$$

most of them reduces to General Relativity, the gravitational *Paradigm* of our time.

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad \text{(Field Equations)}$$

We aim at **the extraction of analytical solutions with physical and mathematical interest.**

Any gravitational field theory is formulated in an action and

$$\boldsymbol{S} = \frac{1}{2\boldsymbol{\kappa}} \int \sqrt{-\boldsymbol{g}} \; [\boldsymbol{R} - 2\boldsymbol{\Lambda}] \; \boldsymbol{dx}^4 + \boldsymbol{\mathcal{S}_M} \qquad \text{(GR)}$$

most of them reduces to General Relativity, the gravitational *Paradigm* of our time.

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad \text{(Field Equations)}$$

We aim at **the extraction of analytical solutions with physical and mathematical interest.**

$$\Rightarrow -\Gamma^{\rho}_{\lambda\mu,\rho} + \Gamma^{\rho}_{\lambda\rho,\mu} - \Gamma^{\kappa}_{\lambda\mu}\Gamma^{\rho}_{\kappa\rho} + \Gamma^{\kappa}_{\lambda\rho}\Gamma^{\rho}_{\kappa\mu} - g_{\mu\nu}\left(\frac{R}{2} - \Lambda\right) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $\Gamma^{\rho}_{\lambda\mu} = \frac{g^{\rho\nu}}{2}(g_{\lambda\nu,\mu} + g_{\mu\nu,\lambda} - g_{\lambda\mu,\nu})$

To overcome the non-linearity and extract a **solution** is required the employment:

Introduction

Any gravitational field theory is formulated in an action and

$$\boldsymbol{S} = \frac{1}{2\boldsymbol{\kappa}} \int \sqrt{-\boldsymbol{g}}\ \ [\boldsymbol{R} - 2\boldsymbol{\Lambda}]\ \ \boldsymbol{dx}^4 + \boldsymbol{\mathcal{S}_M} \tag{GR}$$

most of them reduces to General Relativity, the gravitational *Paradigm* of our time.

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{Field Equations}$$

We aim at **the extraction of analytical solutions with physical and mathematical interest.**

$$\Rightarrow -\Gamma^{\rho}_{\lambda\mu,\rho} + \Gamma^{\rho}_{\lambda\rho,\mu} - \Gamma^{\kappa}_{\lambda\mu}\Gamma^{\rho}_{\kappa\rho} + \Gamma^{\kappa}_{\lambda\rho}\Gamma^{\rho}_{\kappa\mu} - g_{\mu\nu}(\frac{R}{2} - \Lambda) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\bullet \quad \Gamma^{\rho}_{\lambda\mu} = \frac{g^{\rho\nu}}{2}(g_{\lambda\nu,\mu} + g_{\mu\nu,\lambda} - g_{\lambda\mu,\nu})$$

To overcome the non-linearity and extract a **solution** is required the employment:

- An appropriate **formalism** to encode physical or geometric information
- Additional mathematical assumptions, such as **symmetries**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 2(\boldsymbol{\theta}^1\boldsymbol{\theta}^2 - \boldsymbol{\theta}^3\boldsymbol{\theta}^4) \qquad \text{(NP metric)}$$

The metric of NP formalism [1] can be described by a pseudo-orthonormal null basis

$$\boldsymbol{\theta}^1 \equiv n_\mu dx^\mu; \quad \boldsymbol{\theta}^2 \equiv l_\mu dx^\mu; \quad \boldsymbol{\theta}^3 \equiv -\bar{m}_\mu dx^\mu; \quad \boldsymbol{\theta}^4 \equiv -m_\mu dx^\mu; \qquad \underbrace{(l_\mu n^\mu = 1 = -m_\mu \bar{m}^\mu)}_{\textit{Orthogonality Properties}}$$

with directional derivatives (dual basis)

$$D \equiv l^\mu \partial_\mu; \quad \Delta \equiv n^\mu \partial_\mu; \quad \delta \equiv m^\mu \partial_\mu; \quad \bar{\delta} \equiv \bar{m}^\mu \partial_\mu;$$

[1] E.T. Newman, R. Penrose, J. Math. Phys. 3, 566 (1962)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 2(\boldsymbol{\theta}^1\boldsymbol{\theta}^2 - \boldsymbol{\theta}^3\boldsymbol{\theta}^4) \qquad \text{(NP metric)}$$

The metric of NP formalism [1] can be described by a pseudo-orthonormal null basis

$$\boldsymbol{\theta}^1 \equiv n_\mu dx^\mu; \quad \boldsymbol{\theta}^2 \equiv l_\mu dx^\mu; \quad \boldsymbol{\theta}^3 \equiv -\bar{m}_\mu dx^\mu; \quad \boldsymbol{\theta}^4 \equiv -m_\mu dx^\mu; \qquad \underbrace{(l_\mu n^\mu = 1 = -m_\mu \bar{m}^\mu)}_{\textit{Orthogonality Properties}}$$

with directional derivatives (dual basis)

$$D \equiv l^\mu \partial_\mu; \quad \Delta \equiv n^\mu \partial_\mu; \quad \delta \equiv m^\mu \partial_\mu; \quad \bar{\delta} \equiv \bar{m}^\mu \partial_\mu;$$

A derivation on null basis reveals the 1-forms and the 12 complex scalar spin-coefficients

$$d\boldsymbol{\theta}^\alpha = -\Gamma^\alpha_{\mu\nu} \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}^\nu$$

$$\lambda = -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu = -\Gamma_{\mathbf{232}}$$

$$\rho = l_{\mu;\nu} m^\mu \bar{m}^\nu = \Gamma_{\mathbf{142}}$$

$$\pi = -n_{\mu;\nu} \bar{m}^\mu l^\nu = -\Gamma_{\mathbf{234}}$$

$$\kappa = l_{\mu;\nu} m^\mu l^\nu = \Gamma_{\mathbf{144}}$$

$$\beta = \frac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu) = \frac{1}{2}(\Gamma_{\mathbf{341}} - \Gamma_{\mathbf{211}})$$

$$\epsilon = \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu) = \frac{1}{2}(\Gamma_{\mathbf{344}} - \Gamma_{\mathbf{214}})$$

$$\sigma = l_{\mu;\nu} m^\mu m^\nu = \Gamma_{\mathbf{141}}$$

$$\mu = -n_{\mu;\nu} \bar{m}^\mu m^\nu = -\Gamma_{\mathbf{231}}$$

$$\tau = l_{\mu;\nu} m^\mu n^\nu = \Gamma_{\mathbf{143}}$$

$$\nu = -n_{\mu;\nu} \bar{m}^\mu n^\nu = -\Gamma_{\mathbf{233}}$$

$$\alpha = \frac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu) = \frac{1}{2}(\Gamma_{\mathbf{432}} - \Gamma_{\mathbf{122}})$$

$$\gamma = \frac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu) = \frac{1}{2}(\Gamma_{\mathbf{433}} - \Gamma_{\mathbf{123}})$$

[1] E.T. Newman, R. Penrose, J. Math. Phys. 3, 566 (1962)

Commutation Relations (CR) - Lie brackets of the null tetrads

$$[n^\mu, l^\mu] = [\Delta, D] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\tau})\delta - (\bar{\pi} + \tau)\bar{\delta} \tag{CR1}$$

$$[\delta + \bar{\delta}, D] = (\alpha + \bar{\alpha} + \beta + \bar{\beta} - \pi - \bar{\pi})D + (\kappa + \bar{\kappa})\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - (\rho - \epsilon + \bar{\epsilon})\bar{\delta} \tag{CR2$_+$}$$

$$[\delta - \bar{\delta}, D] = (-\alpha + \bar{\alpha} + \beta - \bar{\beta} + \pi - \bar{\pi})D + (\kappa - \bar{\kappa})\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta + (\rho - \epsilon + \bar{\epsilon})\bar{\delta} \tag{CR2$_-$}$$

$$[\delta + \bar{\delta}, \Delta] = -(\nu + \bar{\nu})D + (\tau + \bar{\tau} - \alpha - \bar{\alpha} - \beta - \bar{\beta})\Delta + (\mu - \gamma + \bar{\gamma})\delta + (\bar{\mu} + \gamma - \bar{\gamma})\bar{\delta} \tag{CR3$_+$}$$

$$[\delta - \bar{\delta}, \Delta] = -(\nu - \bar{\nu})D + (\tau - \bar{\tau} + \alpha - \bar{\alpha} - \beta + \bar{\beta})\Delta + (\mu - \gamma + \bar{\gamma})\delta - (\bar{\mu} + \gamma - \bar{\gamma})\bar{\delta} \tag{CR3$_-$}$$

$$[\delta, \bar{\delta}] = -(\mu - \bar{\mu})D - (\rho - \bar{\rho})\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} \tag{CR4}$$

Commutation Relations (CR) - Lie brackets of the null tetrads

$$[n^\mu, l^\mu] = [\Delta, D] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\tau})\delta - (\bar{\pi} + \tau)\bar{\delta} \quad \text{(CR1)}$$

$$[\delta + \bar{\delta}, D] = (\alpha + \bar{\alpha} + \beta + \bar{\beta} - \pi - \bar{\pi})D + (\kappa + \bar{\kappa})\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - (\rho - \epsilon + \bar{\epsilon})\bar{\delta} \quad \text{(CR2}_+\text{)}$$

$$[\delta - \bar{\delta}, D] = (-\alpha + \bar{\alpha} + \beta - \bar{\beta} + \pi - \bar{\pi})D + (\kappa - \bar{\kappa})\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta + (\rho - \epsilon + \bar{\epsilon})\bar{\delta} \quad \text{(CR2}_-\text{)}$$

$$[\delta + \bar{\delta}, \Delta] = -(\nu + \bar{\nu})D + (\tau + \bar{\tau} - \alpha - \bar{\alpha} - \beta - \bar{\beta})\Delta + (\mu - \gamma + \bar{\gamma})\delta + (\bar{\mu} + \gamma - \bar{\gamma})\bar{\delta} \quad \text{(CR3}_+\text{)}$$

$$[\delta - \bar{\delta}, \Delta] = -(\nu - \bar{\nu})D + (\tau - \bar{\tau} + \alpha - \bar{\alpha} - \beta + \bar{\beta})\Delta + (\mu - \gamma + \bar{\gamma})\delta - (\bar{\mu} + \gamma - \bar{\gamma})\bar{\delta} \quad \text{(CR3}_-\text{)}$$

$$[\delta, \bar{\delta}] = -(\mu - \bar{\mu})D - (\rho - \bar{\rho})\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} \quad \text{(CR4)}$$

Weyl Scalars

$\Psi_0 = C_{\kappa\lambda\mu\nu} n^\kappa m^\lambda n^\mu m^\nu$

$\Psi_1 = C_{\kappa\lambda\mu\nu} n^\kappa l^\lambda n^\mu m^\nu$

$\Psi_2 = C_{\kappa\lambda\mu\nu} n^\kappa m^\lambda \bar{m}^\mu l^\nu$

$\Psi_3 = C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu \bar{m}^\nu$

$\Psi_4 = C_{\kappa\lambda\mu\nu} l^\kappa \bar{m}^\lambda l^\mu \bar{m}^\nu$

Real Ricci Scalars

$\Phi_{00} = \frac{1}{2} R_{\mu\nu} n^\mu n^\nu$

$\Phi_{11} = \frac{1}{4} R_{\mu\nu} (n^\mu l^\nu + m^\mu \bar{m}^\nu)$

$\Phi_{22} = \frac{1}{2} R_{\mu\nu} l^\mu l^\nu$

Complex Ricci Scalars

$\Phi_{01} = \frac{1}{2} R_{\mu\nu} n^\mu m^\nu$

$\Phi_{02} = \frac{1}{2} R_{\mu\nu} m^\mu m^\nu$

$\Phi_{12} = \frac{1}{2} R_{\mu\nu} l^\mu m^\nu$

Newman-Penrose Field Equations (NPE) of the formalism in vacuum with Λ

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa\left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta}) - \pi\right] \quad \text{(a)}$$

$$\delta\kappa - D\sigma = -(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon})\sigma + \kappa\left[\tau - \bar{\pi} + 2(\bar{\alpha} + \beta) - (\bar{\alpha} - \beta)\right] - \Psi_o \quad \text{(b)}$$

$$D\tau = \Delta\kappa + \rho(\tau + \bar{\pi}) + \sigma(\pi + \bar{\tau}) + \tau(\epsilon - \bar{\epsilon}) - 2\kappa\gamma - \kappa(\gamma + \bar{\gamma}) + \Psi_1 \quad \text{(c)}$$

$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - 2\nu\epsilon - \nu(\epsilon + \bar{\epsilon}) + \Psi_3 \quad \text{(i)}$$

$$\bar{\delta}\pi - D\lambda = -\pi(\pi + \alpha - \bar{\beta}) - \bar{\sigma}\mu + \nu\bar{\kappa} + \lambda(3\epsilon - \bar{\epsilon}) \quad \text{(g)}$$

$$\delta\tau - \Delta\sigma = \mu\sigma + \bar{\lambda}\rho + \tau(\tau - \bar{\alpha} + \beta) - \sigma(3\gamma - \bar{\gamma}) - \bar{\nu}\kappa \quad \text{(p)}$$

$$D\mu - \delta\pi = \mu\bar{\rho} + \sigma\lambda + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \kappa\nu + \Psi_2 + 2\Lambda \quad \text{(h)}$$

$$\delta\nu - \Delta\mu = \mu(\mu + \gamma + \bar{\gamma}) + \lambda\bar{\lambda} - \bar{\nu}\pi + \nu(\tau - 2(\bar{\alpha} + \beta) + (\bar{\alpha} - \beta)) \quad \text{(n)}$$

$$\Delta\rho - \bar{\delta}\tau = -(\bar{\mu}\rho + \sigma\lambda) - \tau(\bar{\tau} + \alpha - \bar{\beta}) + \nu\kappa + \rho(\gamma + \bar{\gamma}) - \Psi_2 - 2\Lambda \quad \text{(q)}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 \quad \text{(k)}$$

$$\bar{\delta}\mu - \delta\lambda = -\mu(\alpha + \bar{\beta}) - \pi(\mu - \bar{\mu}) - \nu(\rho - \bar{\rho}) - \lambda(\bar{\alpha} - 3\beta) + \Psi_3 \quad \text{(m)}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) \quad \text{(d)}$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \quad \text{(e)}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\epsilon + \rho) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3 \quad \text{(r)}$$

$$-\Delta\beta + \delta\gamma = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) \quad \text{(o)}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \sigma\lambda + \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda \quad \text{(l)}$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \Psi_2 - \Lambda - \kappa\nu + \tau\pi \quad \text{(f)}$$

$$\bar{\delta}\nu - \Delta\lambda = \lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) - \nu\left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta}) + \pi - \bar{\tau}\right] + \Psi_4 \quad \text{(j)}$$

Petrov-Pirani classification is an invariant characterization of the gravitational field[2],[3]

Weyl components	Petrov Types
non-mathematically tractable	Type I
$\Psi_2 \neq 0$	Type D
$\Psi_2\Psi_4 \neq 0$	Type II
Ψ_4 *or* $\Psi_0 \neq 0$	Type N
Ψ_1 *or* $\Psi_3 \neq 0$	Type III

[2] Petrov, A.Z. (1954). Uch. Zapiski Kazan. Gos. Univ. 114 (8): 55–69.
[3] Pirani, F. (1956), On the physical significance of the Riemann tensor, Acta Physica Polonica, 15: 389–405
[4] J. B. Griffiths, and J. Podolský, Cambridge University Press, (2009)

Petrov-Pirani classification is an invariant characterization of the gravitational field[2],[3]

Weyl components	Petrov Types
non-mathematically tractable	Type I
$\Psi_2 \neq 0$	Type D
$\Psi_2\Psi_4 \neq 0$	Type II
Ψ_4 *or* $\Psi_0 \neq 0$	Type N
Ψ_1 *or* $\Psi_3 \neq 0$	Type III

The most known form of type I

$$\Psi_0\Psi_4 \neq (3\Psi_2)^2 \qquad \Psi_0\Psi_4\Psi_2 \neq 0$$

Other versions of type D [4]

$$\Psi_0\Psi_4 = (3\Psi_2)^2 \qquad \Psi_0\Psi_2\Psi_4 \neq 0$$

$$2\Psi_2\Psi_4 = 3\Psi_3^2 \qquad \Psi_2\Psi_3\Psi_4 \neq 0$$

[2] Petrov, A.Z. (1954). Uch. Zapiski Kazan. Gos. Univ. 114 (8): 55–69.

[3] Pirani, F. (1956), On the physical significance of the Riemann tensor, Acta Physica Polonica, 15: 389–405

[4] J. B. Griffiths, and J. Podolský, Cambridge University Press, (2009)

We are interested in spacetimes with hidden symmetries:

- Conjecture of physical significance: There are closed trajectories[5].
- Separability of the Hamilton-Jacobi action resulting in integrable geodesics[6],[7].
- Transformation of the under-determined system of equations (Einstein's Equations, Bianchi Identities) to an over-determined one.

[5] Kruglikov, B. and Matveev, V. S, Nonlinearity, 29,6, 1755 (2016)

[6] Eisenhart, L.P.: Separable systems of Stäckel. Annals of Mathematics, 284–305 (1934)

[7] Kalnins, E.G., Miller, W. Jr: Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations. SIAM Journal on Mathematical Analysis 11(6), 1011–1026 (1980)

We are interested in spacetimes with hidden symmetries:

- Conjecture of physical significance: There are closed trajectories[5].
- Separability of the Hamilton-Jacobi action resulting in integrable geodesics[6],[7].
- Transformation of the under-determined system of equations (Einstein's Equations, Bianchi Identities) to an over-determined one.

This is achievable by assuming the existence of a non-trivial Killing-Stäckel tensor!

$$K_{(\mu\nu;\alpha)} = 0 \qquad \& \qquad \mathcal{K} = K^{\mu\nu} p_\mu p_\nu \tag{1}$$

[5] Kruglikov, B. and Matveev, V. S, Nonlinearity, 29,6, 1755 (2016)

[6] Eisenhart, L.P.: Separable systems of Stäckel. Annals of Mathematics, 284–305 (1934)

[7] Kalnins, E.G., Miller, W. Jr: Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations. SIAM Journal on Mathematical Analysis 11(6), 1011–1026 (1980)

We are interested in spacetimes with hidden symmetries:

- Conjecture of physical significance: There are closed trajectories[5].
- Separability of the Hamilton-Jacobi action resulting in integrable geodesics[6],[7].
- Transformation of the under-determined system of equations (Einstein's Equations, Bianchi Identities) to an over-determined one.

This is achievable by assuming the existence of a non-trivial Killing-Stäckel tensor!

$$K_{(\mu\nu;\alpha)} = 0 \qquad \& \qquad \mathcal{K} = K^{\mu\nu} p_\mu p_\nu \tag{1}$$

The geodesic flow is a Hamiltonian system on the cotangent bundle

$$\bar{m}^2 = g^{\mu\nu} p_\mu p_\nu \qquad \text{(HJ equation)}$$

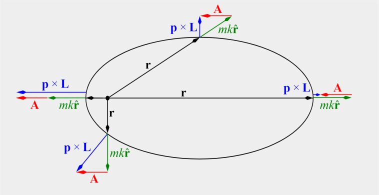
where p_μ are the coordinates or equivalently the canonical momenta.

[5] Kruglikov, B. and Matveev, V. S, Nonlinearity, 29,6, 1755 (2016)

[6] Eisenhart, L.P.: Separable systems of Stäckel. Annals of Mathematics, 284–305 (1934)

[7] Kalnins, E.G., Miller, W. Jr: Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations. SIAM Journal on Mathematical Analysis 11(6), 1011–1026 (1980)

In classical physics the typical example of hidden symmetry is the conserved Laplace-Runge-Lenz vector along a geodesic in Kepler's problem



assuming the existence of a **Killing-Yano** or **Penrose-Floyd Tensor** $f_{\alpha\beta}$ [8]

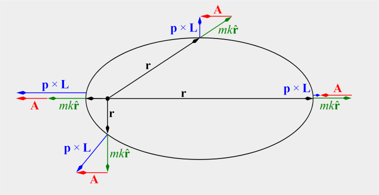
$$K_{\mu\nu} = f_{\mu}{}^{\alpha} f_{\alpha\nu}$$

[8] Frolov, V.P., Krtous, P., Kubiznak, D., Living Rev. Relativ. (2017) 20: 6

[9] T Papakostas. Space-times admitting penrose-floyd tensors. General relativity and gravitation, 17:149–166, 1985.

[10] C D Collinson. Special quadratic first integrals of geodesics. Journal of Physics A: General Physics, 4(6):756, 1971.

In classical physics the typical example of hidden symmetry is the conserved Laplace-Runge-Lenz vector along a geodesic in Kepler's problem



assuming the existence of a **Killing-Yano** or **Penrose-Floyd Tensor** $f_{\alpha\beta}$ [8]

$$K_{\mu\nu} = f_{\mu}{}^{\alpha} f_{\alpha\nu}$$

Algebraically general solutions do not admit Killing-Yano or Penrose-Floyd tensors[9],[10].

[8] Frolov, V.P., Krtous, P., Kubiznak, D., Living Rev. Relativ. (2017) 20: 6

[9] T Papakostas. Space-times admitting penrose-floyd tensors. General relativity and gravitation, 17:149–166, 1985.

[10] C D Collinson. Special quadratic first integrals of geodesics. Journal of Physics A: General Physics, 4(6):756, 1971.

Canonical Killing tensor forms

In canonical coordinates in phase space an integral of motion is described as follows

$$\{\mathcal{K}, H\} \equiv 0$$

The existence of KT provides us with spacetimes with **explicit and hidden symmetries**.

$$\mathcal{K}(x,p) = \underbrace{K^{\mu} p_{\mu}}_{explicit} + \underbrace{K^{\mu\nu} p_{\mu} p_{\nu}}_{hidden} + \underbrace{K^{\mu\nu\sigma} p_{\mu} p_{\nu} p_{\sigma}}_{hidden} + \underbrace{K^{\mu\nu\sigma\rho} p_{\mu} p_{\nu} p_{\sigma} p_{\rho}}_{hidden} + \ldots$$

where the function $\mathcal{K}$ is called polynomial of momenta.

[11] Sadeghian S., Phys. Rev. D, 106, 10 (2002).

Canonical Killing tensor forms

In canonical coordinates in phase space an integral of motion is described as follows

$$\{\mathcal{K}, H\} \equiv 0$$

The existence of KT provides us with spacetimes with **explicit and hidden symmetries**.

$$\mathcal{K}(x,p) = \underbrace{K^{\mu} p_{\mu}}_{explicit} + \underbrace{K^{\mu\nu} p_{\mu} p_{\nu}}_{hidden} + \underbrace{K^{\mu\nu\sigma} p_{\mu} p_{\nu} p_{\sigma}}_{hidden} + \underbrace{K^{\mu\nu\sigma\rho} p_{\mu} p_{\nu} p_{\sigma} p_{\rho}}_{hidden} + \ldots$$

where the function $\mathcal{K}$ is called polynomial of momenta.

The components of $\mathcal{K}$ called Stäckel-Killing tensors and satisfy the **Killing equation**[11].

- Killing vector is a 1-rank Killing tensor and concerns explicit symmetries

$$K_{(\mu;\alpha)} = 0 \tag{KV}$$

- Killing tensor is a 2-rank Killing tensor and concerns hidden symmetries

$$K_{(\mu\nu;\alpha)} = 0 \tag{KT}$$

[11] Sadeghian S., Phys. Rev. D, 106, 10 (2002).

Canonical Forms of Killing Tensor

In line with Churchill [12] we obtained the Canonical forms of Killing tensor [13],

Canonical Forms	Eigenvalues
$K^{\mathbf{0}}_{\mu\nu} = \begin{pmatrix} 0 & \lambda_{\mathbf{1}} & -p & -\bar{p} \\ \lambda_{\mathbf{1}} & 0 & 0 & 0 \\ -p & 0 & \lambda_{\mathbf{7}} & \lambda_{\mathbf{2}} \\ -\bar{p} & 0 & \lambda_{\mathbf{2}} & \lambda_{\mathbf{7}} \end{pmatrix}$	$\lambda_{\mathbf{1}}, \lambda_{\mathbf{1}}, \lambda_{\mathbf{1}}, -(\lambda_{\mathbf{2}} \pm \lambda_{\mathbf{7}})$
$K^{\mathbf{1}}_{\mu\nu} = \begin{pmatrix} \lambda_{\mathbf{0}} & \lambda_{\mathbf{1}} & 0 & 0 \\ \lambda_{\mathbf{1}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\mathbf{7}} & \lambda_{\mathbf{2}} \\ 0 & 0 & \lambda_{\mathbf{2}} & \lambda_{\mathbf{7}} \end{pmatrix}$	$\lambda_{\mathbf{1}}, \lambda_{\mathbf{1}}, -(\lambda_{\mathbf{2}} \pm \lambda_{\mathbf{7}})$
$\Rightarrow \boldsymbol{K^{2}_{\mu\nu}} = \begin{pmatrix} \lambda_{\mathbf{0}} & \lambda_{\mathbf{1}} & 0 & 0 \\ \lambda_{\mathbf{1}} & \lambda_{\mathbf{0}} & 0 & 0 \\ 0 & 0 & \lambda_{\mathbf{7}} & \lambda_{\mathbf{2}} \\ 0 & 0 & \lambda_{\mathbf{2}} & \lambda_{\mathbf{7}} \end{pmatrix}$	$\lambda_{\mathbf{0}} \pm \lambda_{\mathbf{1}}, -(\lambda_{\mathbf{2}} \pm \lambda_{\mathbf{7}})$
$K^{\mathbf{3}}_{\mu\nu} = \begin{pmatrix} \lambda_{\mathbf{0}} & \lambda_{\mathbf{1}} & 0 & 0 \\ \lambda_{\mathbf{1}} & -\lambda_{\mathbf{0}} & 0 & 0 \\ 0 & 0 & \lambda_{\mathbf{7}} & \lambda_{\mathbf{2}} \\ 0 & 0 & \lambda_{\mathbf{2}} & \lambda_{\mathbf{7}} \end{pmatrix}$	$\lambda_{\mathbf{0}} \pm i\lambda_{\mathbf{1}}, -(\lambda_{\mathbf{2}} \pm \lambda_{\mathbf{7}})$

$K^2_{\mu\nu}$ **is the most general canonical Killing tensor form** with four distinct eigenvalues

$$K^2_{\mu\nu} = \lambda_0(\tilde{\theta}^1 \otimes \tilde{\theta}^1 + \tilde{\theta}^2 \otimes \tilde{\theta}^2) + \lambda_1(\tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1) + \lambda_2(\tilde{\theta}^3 \otimes \tilde{\theta}^4 + \tilde{\theta}^4 \otimes \tilde{\theta}^3) + \lambda_7(\tilde{\theta}^3 \otimes \tilde{\theta}^3 + \tilde{\theta}^4 \otimes \tilde{\theta}^4)$$

[12] Churchill R. V., Trans. Amer. Math. Soc. 34, 784 (1932).

[13] **Kokkinos, D., Papakostas, T. The Study of the Canonical forms of Killing tensor in Vacuum with Λ. Gen Relativ Gravit 56, 134 (2024).**

Null tetrads transformations - Implication to the structure

The Structure consists of $g_{\mu\nu}$ and $K^2_{\mu\nu}$

$$K^2_{\mu\nu} = \lambda_0(\tilde{\theta}^1 \otimes \tilde{\theta}^1 + \tilde{\theta}^2 \otimes \tilde{\theta}^2) + \lambda_1(\tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1) + \lambda_2(\tilde{\theta}^3 \otimes \tilde{\theta}^4 + \tilde{\theta}^4 \otimes \tilde{\theta}^3) + \lambda_7(\tilde{\theta}^3 \otimes \tilde{\theta}^3 + \tilde{\theta}^4 \otimes \tilde{\theta}^4)$$

$$g_{\mu\nu} = \tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1 - \tilde{\theta}^3 \otimes \tilde{\theta}^4 - \tilde{\theta}^4 \otimes \tilde{\theta}^3$$

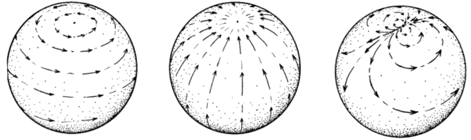
[14] R. Penrose & W. Rindler, Spinors and space-time, Vol 1, Cambridge University Press, 1984

The Structure consists of $g_{\mu\nu}$ and $K^2_{\mu\nu}$

$$K^2_{\mu\nu} = \lambda_0(\tilde{\theta}^1 \otimes \tilde{\theta}^1 + \tilde{\theta}^2 \otimes \tilde{\theta}^2) + \lambda_1(\tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1) + \lambda_2(\tilde{\theta}^3 \otimes \tilde{\theta}^4 + \tilde{\theta}^4 \otimes \tilde{\theta}^3) + \lambda_7(\tilde{\theta}^3 \otimes \tilde{\theta}^3 + \tilde{\theta}^4 \otimes \tilde{\theta}^4)$$

$$g_{\mu\nu} = \tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1 - \tilde{\theta}^3 \otimes \tilde{\theta}^4 - \tilde{\theta}^4 \otimes \tilde{\theta}^3$$

General	Symmetric	Anti-symmetric
$\tilde{\theta}^1 \to e^{-a}(\theta^1 + p\bar{p}\theta^2 + \bar{p}\theta^3 + p\theta^4)$	$\tilde{\theta}^1 \to e\theta^1$	$\tilde{\theta}^1 \to e\theta^2$
$\tilde{\theta}^2 \to e^{a}\ \theta^2$	$\tilde{\theta}^2 \to e\theta^2$	$\tilde{\theta}^2 \to e\theta^1$
$\tilde{\theta}^3 \to e^{-ib}(\theta^3 + p\theta^2)$	$\tilde{\theta}^3 \to e\theta^3$	$\tilde{\theta}^3 \to e\theta^4$
$\tilde{\theta}^4 \to e^{ib}\underbrace{(\theta^4 + \bar{p}\theta^2)}_{Null\ \ Rotation\ \ around\ \ \theta^2}$	$\tilde{\theta}^4 \to e\theta^4$	$\tilde{\theta}^4 \to e\theta^3$



Visualization of the transformations effects on a Riemannian sphere.[14]

[14] R. Penrose & W. Rindler, Spinors and space-time, Vol 1, Cambridge University Press, 1984

Null tetrads transformations - Implication to the structure

The Structure consists of $g_{\mu\nu}$ and $K^2_{\mu\nu}$ with $\lambda_7 = 0$

$$K^2_{\mu\nu} = \lambda_0(\tilde\theta^1 \otimes \tilde\theta^1 + \tilde\theta^2 \otimes \tilde\theta^2) + \lambda_1(\tilde\theta^1 \otimes \tilde\theta^2 + \tilde\theta^2 \otimes \tilde\theta^1) + \lambda_2(\tilde\theta^3 \otimes \tilde\theta^4 + \tilde\theta^4 \otimes \tilde\theta^3)$$

$$g_{\mu\nu} = \tilde\theta^1 \otimes \tilde\theta^2 + \tilde\theta^2 \otimes \tilde\theta^1 - \tilde\theta^3 \otimes \tilde\theta^4 - \tilde\theta^4 \otimes \tilde\theta^3$$

General[15]	Symmetric [16]	Anti-symmetric [17]
$\tilde\theta^1 \to \theta^1$ $\tilde\theta^2 \to \theta^2$ $\tilde\theta^3 \to e^{-ib}\theta^3$ $\tilde\theta^4 \to e^{ib}\theta^4$	$\tilde\theta^1 \to e\theta^1$ $\tilde\theta^2 \to e\theta^2$ $\tilde\theta^3 \to e\theta^3$ $\tilde\theta^4 \to e\theta^4$	$\tilde\theta^1 \to e\theta^2$ $\tilde\theta^2 \to e\theta^1$ $\tilde\theta^3 \to e\theta^4$ $\tilde\theta^4 \to e\theta^3$
$\tilde\sigma = e^{2ib}\sigma, \quad \tilde\lambda = e^{-2ib}\lambda$ $\tilde\kappa = e^{ib}\kappa, \quad \tilde\nu = e^{-ib}\nu$ $\tilde\pi = e^{-ib}\pi, \quad \tilde\tau = e^{ib}\tau$ $\tilde\alpha = e^{-ib}(\alpha + \frac{\bar\delta(ib)}{2})$ $\tilde\beta = e^{ib}(\beta + \frac{\delta(ib)}{2})$ $\tilde\epsilon = \epsilon + \frac{D(ib)}{2}$ $\tilde\gamma = \gamma + \frac{\Delta(ib)}{2}$	$\sigma + e\lambda = 0$ $\kappa + e\nu = 0$ $\pi + e\tau = 0$ $\alpha + e\beta = 0$ $\mu + e\rho = 0$ $\epsilon + e\gamma = 0$	$\sigma + e\bar\lambda = 0$ $\kappa + e\bar\nu = 0$ $\pi + e\bar\tau = 0$ $\alpha + e\bar\beta = 0$ $\mu + e\bar\rho = 0$ $\epsilon + e\bar\gamma = 0$

[15] D. Kokkinos and T. Papakostas, Gen. Relativ. Gravit. 56, 134 (2024).
[16] R. Debever and R. G. McLenaghan, J. Math. Phys. 22, 1711 (1981).
[17] S. R. Czapor and R. G. McLenaghan, J. Math. Phys. 23, 2159 (1982).

Petrov Type D Solution - Characteristics

Our solution admits $K^2_{\mu\nu}$ with $\lambda_7 = 0$ and described by

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\kappa\nu = \tau\pi$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Lambda = 0$$

$$\Psi_0\Psi_4 = 9\Psi_2^2 = const$$

[18] S. Chandrasekhar and B. C. Xanthopoulos. Proc. Royal Soc. London. A. Mathematical and Physical Sciences, 408(1835):175–208, (1986)

Petrov Type D Solution - Characteristics

Our solution admits $K^2_{\mu\nu}$ with $\lambda_7 = 0$ and described by

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\kappa\nu = \tau\pi$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Lambda = 0$$

$$\Psi_0\Psi_4 = 9\Psi_2^2 = const$$

Two null rotations around n^μ and l^μ take us to the **privileged null tetrad frame**[18]

$$\hat{\sigma} = \hat{\lambda} = \hat{\mu} = \hat{\rho} = \hat{\kappa} = \hat{\nu} = \hat{\pi} = \hat{\tau} = 0$$

$$\hat{\Psi}_1 = \hat{\Psi}_3 = \hat{\Psi}_0 = \hat{\Psi}_4 = 0$$

$$\hat{\Psi}_2 = -2\Lambda$$

[18] S. Chandrasekhar and B. C. Xanthopoulos. Proc. Royal Soc. London. A. Mathematical and Physical Sciences, 408(1835):175–208, (1986)

Our solution admits $K^2_{\mu\nu}$ with $\lambda_7 = 0$ and described by

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\kappa\nu = \tau\pi$$
$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Lambda = 0$$
$$\Psi_0\Psi_4 = 9\Psi_2^2 = const$$

Two null rotations around n^μ and l^μ take us to the **privileged null tetrad frame**

$$\underbrace{\hat{\sigma} = \hat{\lambda}}_{shearless} = \underbrace{\hat{\mu} = \hat{\rho}}_{non-diverging} = \underbrace{\hat{\kappa} = \hat{\nu}}_{geodesic} = \hat{\pi} = \hat{\tau} = 0$$
$$\hat{\Psi}_1 = \hat{\Psi}_3 = \hat{\Psi}_0 = \hat{\Psi}_4 = 0$$
$$\hat{\Psi}_2 = -2\Lambda$$

[19] R. Debever, BulL Soc. Math. Belg. XXIII, 360 (1971)
[20] J. F. Plebański, and M. Demiański, Ann. Phys., 98 (1976)

Petrov Type D Solution - Characteristics

Our solution admits $K^2_{\mu\nu}$ with $\lambda_7 = 0$ and described by

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\kappa\nu = \tau\pi$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Lambda = 0$$

$$\Psi_0\Psi_4 = 9\Psi_2^2 = const$$

Two null rotations around n^μ and l^μ take us to the **privileged null tetrad frame**

$$\underbrace{\hat{\sigma} \ = \ \hat{\lambda}}_{shearless} = \underbrace{\hat{\mu} \ = \ \hat{\rho}}_{non-diverging} = \underbrace{\hat{\kappa} \ = \ \hat{\nu}}_{geodesic} = \hat{\pi} = \hat{\tau} = 0$$

$$\hat{\Psi}_1 = \hat{\Psi}_3 = \hat{\Psi}_0 = \hat{\Psi}_4 = 0$$

$$\hat{\Psi}_2 = -2\Lambda$$

Our solution belongs to Debever and Plebański-Demiański family (Vacuum, $\Lambda \neq 0$),[19], [20]
Goldberg-Sachs theorem does hold since $\hat{n}^\mu, \hat{l}^\mu$ are geodesic and shearless.

[19] R. Debever, BulL Soc. Math. Belg. XXIII, 360 (1971)

[20] J. F. Plebański, and M. Demiański, Ann. Phys., 98 (1976)

All spacetimes with constant curvature can be expressed as [21] [22]

$$ds^2 = \Omega_1 \left[\Sigma^2(x_1, g_1)dt^2 - dx_1{}^2\right] - \Omega_2 \left[\Sigma^2(x_2, g_2)dz^2 + dx_2{}^2\right] \qquad \text{(General Metric)}$$

where $\Sigma^2(x_i, g_i)$ is represented by $\sin(x_i)$ or $\sinh(x_i)$ with $g_i = -1, 0, +1$ and $\Omega_i = const$.

[21] H. Stephani, D. Kramer, et.al, Exact Solutions of Einstein's Field Equations, Cambridge University Press, (2003)
[22] Schmidt, B.G., Gen. Rel. Grav., 2, 105 (1971)
[23] Kokkinos D. and Papakostas T. Carter's case [D] admits the second canonical form of the Killing tensor (2023) (Unpublished)

Petrov type D solution - Cosmological models with constant curvature

All spacetimes with constant curvature can be expressed as [21] [22]

$$ds^2 = \Omega_1 \left[\Sigma^2(x_1, g_1)dt^2 - d{x_1}^2\right] - \Omega_2 \left[\Sigma^2(x_2, g_2)dz^2 + d{x_2}^2\right] \qquad \text{(General Metric)}$$

where $\Sigma^2(x_i, g_i)$ is represented by $\sin(x_i)$ or $\sinh(x_i)$ with $g_i = -1, 0, +1$ and $\Omega_i = const.$

Our spacetimes describe cosmological models with radius $R_c = \frac{1}{\sqrt{6\Lambda}} = 13.654$ billion lys

$$ds^2 = \frac{1}{6\Lambda}\left[\sin^2(x)Cdt^2 - dx^2 - \sin^2(y)Cdz^2 - dy^2\right] \qquad (\text{Nariai } \Lambda > 0;\ C > 0)$$

$$ds^2 = \frac{1}{6\Lambda}\left[\sinh^2(x)Cdt^2 - dx^2 - \sinh^2(y)Cdz^2 - dy^2\right] \qquad (\text{anti-Nariai } \Lambda < 0;\ C > 0)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$ds^2 = \frac{1}{6\Lambda}\left[\cosh^2(x)|C|dt^2 - dx^2 - \cosh^2(y)|C|dz^2 - dy^2\right] \qquad (\text{anti-Nariai } \Lambda < 0;\ C < 0)$$

it can be easily transformed to **de Sitter**, **Carter's Case [D]**[23], chargeless **Robinson-Bertotti** universe.

---

[21] H. Stephani, D. Kramer, et.al, Exact Solutions of Einstein's Field Equations, Cambridge University Press, (2003)

[22] Schmidt, B.G., Gen. Rel. Grav., 2, 105 (1971)

[23] Kokkinos D. and Papakostas T. Carter's case [D] admits the second canonical form of the Killing tensor (2023) (Unpublished)

Petrov type D solution - Killing tensor and Hidden symmetry

The Killing tensor of spacetime with $\Lambda > 0$; $C > 0$ is given

$$K^{2\mu}{}_{\nu} = \begin{pmatrix} \lambda_0 + \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_0 - \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

$$\lambda_0 + \lambda_1 = \lambda_+ \frac{C}{6\Lambda} sin^2(x) - \lambda_2$$
$$\lambda_0 - \lambda_1 = \lambda_- \frac{C}{6\Lambda} sin^2(y) + \lambda_2$$
$$\lambda_2 = const$$

The Killing tensor of spacetime with $\Lambda > 0$; $C > 0$ is given

$$K^{2\mu}{}_{\nu} = \begin{pmatrix} \lambda_0 + \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_0 - \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

$$\lambda_0 + \lambda_1 = \lambda_+ \frac{C}{6\Lambda} sin^2(x) - \lambda_2$$
$$\lambda_0 - \lambda_1 = \lambda_- \frac{C}{6\Lambda} sin^2(y) + \lambda_2$$
$$\lambda_2 = const$$

The general expression of the hidden symmetry of our type D solution is given

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu =$$

Petrov type D solution - Killing tensor and Hidden symmetry

The Killing tensor of spacetime with $\Lambda > 0$; $C > 0$ is given

$$K^{2\mu}{}_{\nu} = \begin{pmatrix} \lambda_0 + \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_0 - \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

$$\lambda_0 + \lambda_1 = \lambda_+ \frac{C}{6\Lambda} sin^2(x) - \lambda_2$$
$$\lambda_0 - \lambda_1 = \lambda_- \frac{C}{6\Lambda} sin^2(y) + \lambda_2$$
$$\lambda_2 = const$$

The general expression of the hidden symmetry of our type D solution is given

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu = \frac{1}{2} \left[\frac{(E+L)^2}{\lambda_- \frac{C}{6\Lambda} sin^2(y) + \lambda_2} + \frac{(E-L)^2}{\lambda_+ \frac{C}{6\Lambda} sin^2(x) - \lambda_2} \right] - \frac{1}{6|\Lambda|\lambda_2} \sqrt{\left(\frac{E^2}{\frac{C}{6\Lambda} sin^2(x)} - \mathcal{K}_+ \right) \left(\mathcal{K} - \frac{L^2}{\frac{C}{6\Lambda} sin^2(y)} \right)}$$

where by equations of motion we get the canonical momenta to be

$$p_\mu = \left(E, -L, \frac{1}{\sqrt{12|\Lambda|}} \left[\frac{E^2}{\frac{C}{6\Lambda} \sin^2(x)} - \mathcal{K} - 2\bar{m} \right]^{1/2}, \frac{1}{\sqrt{12|\Lambda|}} \left[\mathcal{K} - \frac{L^2}{\frac{C}{6\Lambda} sin^2(y)} \right]^{1/2} \right)$$

Choosing now $y = \pi/2$ (Azimuthal plane) we get the hidden symmetry

$$\mathcal{K} = L^2 \frac{6\Lambda}{C}$$

which is analogous to L^2 as exprected by the Carter's constant.

Petrov type I solution- Characteristics and consideration of $\Lambda < 0$

The characteristics of our type I solution are given

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Psi_2^* = 0$$

$$\Psi_0 \Psi_4 \neq 9\Psi_2^2$$

Petrov type I solution- Characteristics and consideration of $\Lambda < 0$

The characteristics of our type I solution are given

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Psi_2^* = 0$$

$$\Psi_0 \Psi_4 \neq 9\Psi_2^2$$

At a first glance our metric for $\Lambda < 0$, can be considered as a time-dependent generalization of

$$ds^2 = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[\Pi_1 x + \Pi_2 \frac{V_1 x + y}{1 - V_1}\right]\right)}{\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[K_1 x + K_2 \frac{V_1 x + y}{1 - V_1}\right]}\left[[A(t)dt + dz]^2 - \left[K_1 x + K_2 \frac{V_1 x + y}{1 - V_1}\right]^2 [B(t)dt + dz]^2\right]$$

$$- \frac{(\frac{dx}{2})^2 + (\frac{V_1 dx + dy}{21 - V_1})^2}{\cosh^2\left(\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[\Pi_1 x + \Pi_2 \frac{V_1 x + y}{1 - V_1}\right]\right)}$$

Petrov type I solution- Characteristics and consideration of $\Lambda < 0$

The characteristics of our type I solution are given

$$\sigma = \lambda = \mu = \rho = \bar{\alpha} + \beta = \kappa + \bar{\nu} = \pi + \bar{\tau} = \epsilon = \gamma = 0$$

$$\Psi_1 = \Psi_3 = \Psi_0 - \Psi_4^* = \Psi_2 - \Psi_2^* = 0$$

$$\Psi_0 \Psi_4 \neq 9\Psi_2^2$$

At a first glance our metric for $\Lambda < 0$, can be considered as a time-dependent generalization of

$$ds^2 = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[\Pi_1 x + \Pi_2 \frac{V_1 x + y}{1 - V_1}\right]\right)}{\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[K_1 x + K_2 \frac{V_1 x + y}{1 - V_1}\right]}\left[[A(t)dt + dz]^2 - \left[K_1 x + K_2 \frac{V_1 x + y}{1 - V_1}\right]^2 [B(t)dt + dz]^2\right]$$
$$- \frac{(\frac{dx}{2})^2 + (\frac{V_1 dx + dy}{21 - V_1})^2}{\cosh^2\left(\frac{1}{2}\sqrt{\frac{6|\Lambda|}{\Pi\bar{\Pi}}}\left[\Pi_1 x + \Pi_2 \frac{V_1 x + y}{1 - V_1}\right]\right)}$$

the **general stationary axisymmetric spacetime** where ψ, ω, x depend on r, z alone

$$ds^2 = e^{2\psi}(dt + \alpha d\phi)^2 - e^{2\omega} d\phi^2 - e^{2x}(dr^2 + dz^2) \quad (1)$$

or as a generalization of the **non-static cylindrical line element**

$$ds^2 = -e^{2\psi}(dz + \alpha d\phi)^2 - e^{2\omega} d\phi^2 - e^{2x}(dr^2 - dt^2) \quad (2)$$

which is obtained by $t \to iz$; $z \to it$; $\alpha \to i\alpha$

Petrov type I solution - A brief analysis of Weyl scalars

Taking a closer look we can set $V_1 = 0$; $\Pi_1 = \Pi_2$; $K_1 = K_2$ and implying

$$z \to \phi; \qquad \frac{x+y}{2} \to r; \qquad \frac{x-y}{2} \to z$$

with ranges $t, z \in (-\infty, +\infty)$, $r \in (0, +\infty)$, $\phi \in [0, 2\pi)$ our metric results in

$$ds^2 = \frac{|\Pi_1| \tanh\left(\sqrt{6|\Lambda|}r\right)}{|K_1|\sqrt{6|\Lambda|}r} \left[\underbrace{\left[A^2(t) - (2K_1 r)^2 B^2(t)\right]}_{g_{tt}} dt^2 + 2 \underbrace{\left[A(t) - (2K_1 r)^2 B(t)\right]}_{g_{t\phi}} dt d\phi - \underbrace{\left[(2K_1 r)^2 - 1\right]}_{g_{\phi\phi}} d\phi^2 \right] - \frac{dr^2 + dz^2}{2\cosh^2\left(\sqrt{6|\Lambda|}r\right)}$$

Petrov type I solution - A brief analysis of Weyl scalars

Taking a closer look we can set $V_1 = 0$; $\Pi_1 = \Pi_2$; $K_1 = K_2$ and implying

$$z \to \phi; \qquad \frac{x+y}{2} \to r; \qquad \frac{x-y}{2} \to z$$

with ranges $t, z \in (-\infty, +\infty), r \in (0, +\infty), \phi \in [0, 2\pi)$ our metric results in

$$ds^2 = \frac{|\Pi_1| \tanh\left(\sqrt{6|\Lambda|}r\right)}{|K_1|\sqrt{6|\Lambda|}r} \left[\underbrace{\left[A^2(t) - (2K_1 r)^2 B^2(t)\right]}_{g_{tt}} dt^2 + 2 \underbrace{\left[A(t) - (2K_1 r)^2 B(t)\right]}_{g_{t\phi}} dt d\phi - \underbrace{\left[(2K_1 r)^2 - 1\right]}_{g_{\phi\phi}} d\phi^2 \right] - \frac{dr^2 + dz^2}{2\cosh^2\left(\sqrt{6|\Lambda|}r\right)}$$

$$2\Psi_2 = -2|\Lambda| - \frac{6|\Lambda|}{\tanh^2 \sqrt{6|\Lambda|}r} - \frac{\cosh^2 \sqrt{6|\Lambda|}r}{4r^2} \quad \underset{r \to 0^+, +\infty}{\longrightarrow} \quad \Psi_2 \to -\infty \qquad \text{(Curvature Singularity)}$$

$$\Psi_0 \Psi_4 = \frac{3|\Lambda| \sinh^2 (2\sqrt{6|\Lambda|}r)}{8r^2} \quad \underset{r \to +\infty}{\longrightarrow} \quad \Psi_0\Psi_4 \to +\infty$$

Taking a closer look we can set $V_1 = 0$; $\Pi_1 = \Pi_2$; $K_1 = K_2$ and implying

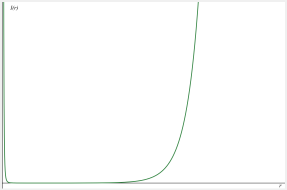
$$z \to \phi; \qquad \frac{x+y}{2} \to r; \qquad \frac{x-y}{2} \to z$$

with ranges $t, z \in (-\infty, +\infty)$, $r \in (0, +\infty)$, $\tilde{\phi} \in [0, 2\pi)$ our metric results in

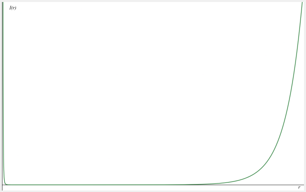
$$ds^2 = \frac{|\Pi_1| \tanh\left(\sqrt{6|\Lambda|}r\right)}{|K_1|\sqrt{6|\Lambda|}r} \left[\underbrace{\left[A^2(t) - (2K_1 r)^2 B^2(t)\right]}_{g_{tt}} dt^2 + 2 \underbrace{\left[A(t) - (2K_1 r)^2 B(t)\right]}_{g_{t\phi}} dt d\phi - \underbrace{\left[(2K_1 r)^2 - 1\right]}_{g_{\phi\phi}} d\phi^2 \right] - \frac{dr^2 + dz^2}{2\cosh^2\left(\sqrt{6|\Lambda|}r\right)}$$

To determine the curvature singularities we have to consider **Kretschmann scalar**

$$\mathcal{K} \propto \mathcal{R}e\{I(r)\} = \Psi_0\Psi_4 + 3\Psi_2^2$$



$|\Lambda| = 0.1\ m^{-2}$



$|\Lambda| = 0.05\ m^{-2}$

A brief Analysis - Conformal factor and reduction in vacuum

We can introduce a regular conformal factor $\Omega^2(r)$ and its limit to $r \to +\infty$ indicating the asymptotically flat character of our spacetime.

$$ds^2 = \underbrace{\frac{|\Pi_1|}{|K_1|} \frac{\tanh(\sqrt{6\Lambda} r)}{\sqrt{6\Lambda} r}}_{\Omega^2(r)} \left[[A(t)dt + d\phi]^2 - (2K_1 r)^2 [B(t)dt + d\phi]^2 - \frac{|K_1|\sqrt{6\Lambda} r}{|\Pi_1|} \frac{dr^2 + dz^2}{\sinh\left(2\sqrt{6|\Lambda|} r\right)} \right]$$

[24] N. Ozdemir, Int.J.Mod.Phys.A 20 (2005) 2821-2832

A brief Analysis - Conformal factor and reduction in vacuum

We can introduce a regular conformal factor $\Omega^2(r)$ and its limit to $r \to +\infty$ indicating the asymptotically flat character of our spacetime.

$$ds^2 = \underbrace{\frac{|\Pi_1|}{|K_1|}\frac{\tanh(\sqrt{6\Lambda}r)}{\sqrt{6\Lambda}r}}_{\Omega^2(r)}\left[[A(t)dt + d\phi]^2 - (2K_1 r)^2[B(t)dt + d\phi]^2 - \frac{|K_1|\sqrt{6\Lambda}r}{|\Pi_1|}\frac{dr^2 + dz^2}{\sinh\left(2\sqrt{6|\Lambda|}r\right)}\right]$$

Limitation to Vacuum ($\Lambda \to 0$)

$$ds^2 = \frac{|\Pi_1|}{|K_1|}\left[\underbrace{\left[A^2(t) - (2K_1 r)^2 B^2(t)\right]}_{g_{tt}} dt^2 + d\phi + 2\underbrace{\left[A(t) - (2K_1 r)^2 B(t)\right]}_{g_{t\phi}} dt d\phi - \underbrace{\left[(2K_1 r)^2 - 1\right]}_{g_{\phi\phi}} d\phi^2\right] - \left(dr^2 + dz^2\right)$$

Considering[24] the mass per length to be equal to $m = \frac{2|K_1|-1}{4} \approx 10^{22}\,\frac{m}{l}$ the signature preservation dictates that

$$r > \frac{1}{2|K_1|} \approx 10^{-24} \quad l$$

At last, in vacuum limit it reduces to a **non-stationary rotating cosmic string-like with curvature singularity**.

$$\lim_{\Lambda\to 0}\Psi_2 \to -\frac{5}{8r^2} \qquad\qquad \lim_{\Lambda\to 0}\Psi_0\Psi_4 \to 0$$

[24] N. Ozdemir, Int.J.Mod.Phys.A 20 (2005) 2821-2832

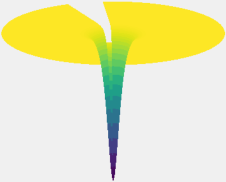
Summary & Conclusions

- We start a conversation about the preferability of null tetrads transformations upon a structure ($g_{\mu\nu}$, $K_{\mu\nu}$).
- Employing the most general null tetrads transformation we proved[25] that:
  - The **null rotation** is not applicable to a structure which incorporates the Canonical Killing tensor forms.
  - Applies **constraints on Killing tensors**.
  - The implication of the general null tetrad transformation results in either a boost or a spatial rotation and it produces **algebraically special solutions** only.
- Assuming the $K^2_{\mu\nu}$ with $\lambda_7 = 0$ we obtained a Petrov type D solutions which describes a general family of spacetimes with constant curvature regarding **Nariai** and **anti-Nariai**,**Robinson-Bertotti**, **Carter's Case [D]**, **de-Sitter and anti-de-Sitter** in vacuum with Λ.
- On the other hand, assuming again the $K^2_{\mu\nu}$ with $\lambda_7 = 0$ we extract a **Petrov type I solution with the exact same spin coefficients** employing the antisymmetric null tetrad transformation.
- This spacetimes results in **an asymptotically flat rotating cosmic string-like spacetime with a curvature singularity at the axis**.

---

[25] D. Kokkinos and T. Papakostas Gen. Relativ. Gravit. 56, 134 (2024)

Thank you for your Attention!



Additional Slides - Killing equations

$$D\lambda_0 = 2\lambda_0(\epsilon + \bar{\epsilon}), \tag{3}$$

$$\Delta\lambda_0 = -2\lambda_0(\gamma + \bar{\gamma}), \tag{4}$$

$$\delta\lambda_0 = 2\left[\lambda_0(\bar{\alpha} + \beta + \bar{\pi}) - \kappa(\lambda_1 + \lambda_2) - \bar{\kappa}\lambda_7\right], \tag{5}$$

$$\delta\lambda_0 = 2\left[-\lambda_0(\bar{\alpha} + \beta + \tau) + q\bar{\nu}(\lambda_1 + \lambda_2) + q\nu\lambda_7\right], \tag{6}$$

$$D\lambda_1 = 2\lambda_0(\gamma + \bar{\gamma}), \tag{7}$$

$$\Delta\lambda_1 = -2q\lambda_0(\epsilon + \bar{\epsilon}), \tag{8}$$

$$\delta\lambda_1 = -q\lambda_0(\kappa - q\bar{\nu}) + (\lambda_1 + \lambda_2)(\bar{\pi} - \tau) + \lambda_7(\pi - \bar{\tau}), \tag{9}$$

$$D\lambda_2 = \lambda_0(\mu + \bar{\mu}) - (\lambda_1 + \lambda_2)(\rho + \bar{\rho}) - \lambda_7(\sigma + \bar{\sigma}), \tag{10}$$

$$\Delta\lambda_2 = -q\lambda_0(\rho + \bar{\rho}) + (\lambda_1 + \lambda_2)(\mu + \bar{\mu}) + \lambda_7(\lambda + \bar{\lambda}), \tag{11}$$

$$\delta\lambda_2 = 2(\alpha - \bar{\beta})\lambda_7, \tag{12}$$

$$D\lambda_7 = 2\left[\lambda_0\lambda - (\lambda_1 + \lambda_2)\bar{\sigma} - \lambda_7(\rho + \epsilon - \bar{\epsilon})\right], \tag{13}$$

$$\Delta\lambda_7 = -2\left[q\lambda_0\bar{\sigma} - (\lambda_1 + \lambda_2)\lambda + \lambda_7(\gamma - \bar{\gamma} - \bar{\mu})\right], \tag{14}$$

$$\delta\lambda_7 = -2\lambda_7(\alpha - \bar{\beta}). \tag{15}$$

Additional Slides - Key equations

$$\Psi_2 - \Lambda = \kappa\nu - \tau\pi, \tag{i}$$

$$\Psi_1 = \kappa\mu - \sigma\pi, \tag{ii}$$

$$\Psi_2 - \Lambda = \mu\rho - \sigma\lambda, \tag{iii}$$

$$\mu\tau - \sigma\nu = 0. \tag{iv}$$

Table: $\rho - \bar{\rho} = 0 = \rho + \bar{\rho}$

$\rho = 0 \neq \Psi_2$ Type D	$\rho + \bar{\rho} = 0 = \Psi_2$ Type N	$\rho + \bar{\rho} = 0 = \Psi_2$ Type N
$\kappa\nu = \pi\tau$ $\Psi_0\Psi_4 = 9{\Psi_2}^2$ $\Psi_1 = \Psi_3 = 0$	$\nu = 0 = \pi\tau$ $\Psi_0 \neq 0$ $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$	$\kappa = 0 = \pi\tau$ $\Psi_4 \neq 0$ $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$
$d\lambda_2 = 0$	$d\lambda_2 = 0$	$d\lambda_2 = 0$

Table: $\rho - \bar{\rho} = 0 \neq \rho + \bar{\rho}$

Type N	Type N	Type N	Type N	Type N	Type N
$\nu = 0 = \tau$ $\Psi_0 \neq 0$	$\nu = 0 = \pi$ $\Psi_4 \neq 0$	$\kappa = 0 = \tau$ $\Psi_0 \neq 0$	$\kappa = 0 = \pi$ $\Psi_4 \neq 0$	$\nu = \pi = \tau = 0$ $\Psi_0 \neq 0$	$\kappa = \pi = \tau = 0$ $\Psi_4 \neq 0$
$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$