Black Hole Ringdown in an Astrophysical Environment

Petr Kotlařík

based on joint work with Che-Yu Chen, arXiv:2307.07360

Astronomical Institute of the Czech Academy of Sciences

2 September 2025, NEB-21 Conference, Corfu





Quasinormal modes

ringing a black hole

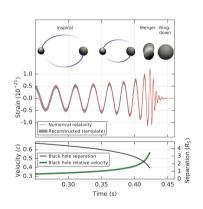
- \longrightarrow field dynamics \Rightarrow the whole system is dissipative
- \longrightarrow exponential decay of gravitational waves
- → sum of exponentially damped sinusoids
 ≡ quasinormal modes



Quasinormal modes

ringing a black hole

- \longrightarrow field dynamics \Rightarrow the whole system is dissipative
- \longrightarrow exponential decay of gravitational waves
- \longrightarrow sum of exponentially damped sinusoids \equiv quasinormal modes



Isolated vacuum black hole \longrightarrow QNM spectrum is uniquely determined by the black hole's mass and spin (and/or charge).

Black hole spectroscopy is a powerful tool for **testing general relativity** (imprints of new physics?):

- modified theories of gravity
- quantum gravity effects
- testing dark matter models
- challenging the black hole paradigm: exotic compact objects



Isolated vacuum black hole \longrightarrow QNM spectrum is uniquely determined by the black hole's mass and spin (and/or charge).

Black hole spectroscopy is a powerful tool for **testing general relativity** (imprints of new physics?):

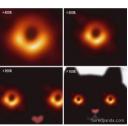
- modified theories of gravity
- quantum gravity effects
- testing dark matter models
- challenging the black hole paradigm: exotic compact objects



Isolated vacuum black hole \longrightarrow QNM spectrum is uniquely determined by the black hole's mass and spin (and/or charge).

Black hole spectroscopy is a powerful tool for **testing general relativity** (imprints of new physics?):

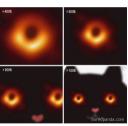
- modified theories of gravity
- quantum gravity effects
- testing dark matter models
- challenging the black hole paradigm: exotic compact objects



Isolated vacuum black hole \longrightarrow QNM spectrum is uniquely determined by the black hole's mass and spin (and/or charge).

Black hole spectroscopy is a powerful tool for **testing general relativity** (imprints of new physics?):

- modified theories of gravity
- quantum gravity effects
- testing dark matter models
- challenging the black hole paradigm: exotic compact objects



Most studies focus on the **static and spherically symmetric case**:a Schwarzschild black hole surrounded by a (dark-)matter halo or dust shells.

(Barausse, Cardoso, and Pani, 2014; Cheung et al., 2022; Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025).

- Certain small changes to the system lead to instabilities in the QNM spectrum (Cheung et al., 2022).
- The **fundamental mode** can be extracted from time-domain simulations (Berti et al., 2022), but the role of **overtones** remains uncertain.
- Even after a recent systematic investigation by Cardoso, Kastha, and Macedo (2024), the **physical relevance** of QNM spectral instabilities remains **unresolved**.

Most studies focus on the **static and spherically symmetric case**:a Schwarzschild black hole surrounded by a (dark-)matter halo or dust shells.

(Barausse, Cardoso, and Pani, 2014; Cheung et al., 2022; Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025).

- Certain small changes to the system lead to instabilities in the QNM spectrum (Cheung et al., 2022).
- The **fundamental mode** can be extracted from time-domain simulations (Berti et al., 2022), but the role of **overtones** remains uncertain.
- Even after a recent systematic investigation by Cardoso, Kastha, and Macedo (2024), the **physical relevance** of QNM spectral instabilities remains **unresolved**.

Most studies focus on the **static and spherically symmetric case**:a Schwarzschild black hole surrounded by a (dark-)matter halo or dust shells.

(Barausse, Cardoso, and Pani, 2014; Cheung et al., 2022; Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025).

- Certain small changes to the system lead to instabilities in the QNM spectrum (Cheung et al., 2022).
- The **fundamental mode** can be extracted from time-domain simulations (Berti et al., 2022), but the role of **overtones** remains uncertain.
- Even after a recent systematic investigation by Cardoso, Kastha, and Macedo (2024), the **physical relevance** of QNM spectral instabilities remains **unresolved**.

Most studies focus on the **static and spherically symmetric case**:a Schwarzschild black hole surrounded by a (dark-)matter halo or dust shells.

(Barausse, Cardoso, and Pani, 2014; Cheung et al., 2022; Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025).

- Certain small changes to the system lead to instabilities in the QNM spectrum (Cheung et al., 2022).
- The **fundamental mode** can be extracted from time-domain simulations (Berti et al., 2022), but the role of **overtones** remains uncertain.
- Even after a recent systematic investigation by Cardoso, Kastha, and Macedo (2024), the **physical relevance** of QNM spectral instabilities remains **unresolved**.

Accretion disk

This talk: Accretion disk as a natural astrophysical environment of a black hole

→ departing from the spherical symmetry.



Figure 1: Interstellar black hole (James et al., 2015).

Outline

1 The Schwarzschild + disk model (SBH+disk)

QNMs of a scalar field

3 Particular results

Table of Contents

2 QNMs of a scalar field

Particular results

Weyl metric

Gravitational field of static and axially symmetric vacuum spacetimes is described by

$$ds^{2} = -e^{2\nu} dt^{2} + \rho^{2} e^{-2\nu} d\phi^{2} + e^{2\lambda - 2\nu} (d\rho^{2} + dz^{2}), \qquad (1)$$

where t, ρ, ϕ, z are the **Weyl** cylindrical **coordinates** and $\nu(\rho, z), \lambda(\rho, z)$.

Vacuum (outside of sources) Einstein equations

$$\nabla \cdot (\nabla \nu) = 0 \tag{2}$$

$$\lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2) \tag{3}$$

$$\lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z} \tag{4}$$

Weyl metric

Gravitational field of static and axially symmetric vacuum spacetimes is described by

$$ds^{2} = -e^{2\nu} dt^{2} + \rho^{2} e^{-2\nu} d\phi^{2} + e^{2\lambda - 2\nu} (d\rho^{2} + dz^{2}), \qquad (1)$$

where t, ρ, ϕ, z are the **Weyl** cylindrical **coordinates** and $\nu(\rho, z), \lambda(\rho, z)$.

Vacuum (outside of sources) Einstein equations:

$$\nabla \cdot (\nabla \nu) = 0 \tag{2}$$

$$\lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2) \tag{3}$$

$$\lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z} \tag{4}$$

Weyl metric

Gravitational field of static and axially symmetric vacuum spacetimes is described by

$$ds^{2} = -e^{2\nu} dt^{2} + \rho^{2} e^{-2\nu} d\phi^{2} + e^{2\lambda - 2\nu} (d\rho^{2} + dz^{2}), \qquad (1)$$

where t, ρ, ϕ, z are the **Weyl** cylindrical **coordinates** and $\nu(\rho, z), \lambda(\rho, z)$.

Vacuum (outside of sources) Einstein equations:

$$\nabla \cdot (\nabla \nu) = 0$$
 ... look, Laplace equation! (2)

$$\lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2) \tag{3}$$

$$\lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z} \tag{4}$$



Multiple sources in GR

Consider **two** distinct **solutions**, described respectively by ν_1 , ν_2 , and λ_1 , λ_2 respectively

Their common gravitational field is given by

$$\nu = \nu_1 + \nu_2 \,, \tag{5}$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{int} \,, \tag{6}$$

where λ_{int} satisfies

$$\lambda_{\text{int},\rho} = 2\rho(\nu_{\text{Schw},\rho}\nu_{\text{disk},\rho} - \nu_{\text{Schw},z}\nu_{\text{disk},z}), \tag{7}$$

$$\lambda_{\text{int},z} = 2\rho(\nu_{\text{Schw},\rho}\nu_{\text{disk},z} + \nu_{\text{Schw},z}\nu_{\text{disk},\rho}). \tag{8}$$

Multiple sources in GR

Consider **two** distinct **solutions**, described respectively by ν_1 , ν_2 , and λ_1 , λ_2 respectively

Their common gravitational field is given by

$$\nu = \nu_1 + \nu_2 \,, \tag{5}$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{int} \,, \tag{6}$$

where λ_{int} satisfies

$$\lambda_{\text{int},\rho} = 2\rho(\nu_{\text{Schw},\rho}\nu_{\text{disk},\rho} - \nu_{\text{Schw},z}\nu_{\text{disk},z}), \tag{7}$$

$$\lambda_{\text{int},z} = 2\rho(\nu_{\text{Schw},\rho}\nu_{\text{disk},z} + \nu_{\text{Schw},z}\nu_{\text{disk},\rho}). \tag{8}$$

SBH+disk model

Schwarzschild black hole in Weyl coordinates

$$u_{\mathsf{Schw}} = \frac{1}{2} \ln \left(\frac{R_{+} + R_{-} - 2M}{R_{+} + R_{-} + 2M} \right) , \qquad \lambda_{\mathsf{Schw}} = \frac{1}{2} \ln \left[\frac{(R_{+} + R_{-})^{2} - 4M^{2}}{4R_{+}R_{-}} \right] , \qquad (9)$$

where

$$R_{\pm} = \sqrt{\rho^2 + (|z| \mp M)^2} \,. \tag{10}$$

Add a suitable disk solution with a "reasonable" density profile and integrate λ_{int} .

SBH+disk model

Schwarzschild black hole in Weyl coordinates

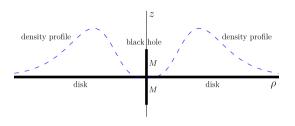
$$u_{\mathsf{Schw}} = \frac{1}{2} \ln \left(\frac{R_{+} + R_{-} - 2M}{R_{+} + R_{-} + 2M} \right) , \qquad \lambda_{\mathsf{Schw}} = \frac{1}{2} \ln \left[\frac{(R_{+} + R_{-})^{2} - 4M^{2}}{4R_{+}R_{-}} \right] , \qquad (9)$$

where

$$R_{\pm} = \sqrt{\rho^2 + (|z| \mp M)^2} \,. \tag{10}$$

Add a suitable disk solution with a "reasonable" density profile and integrate λ_{int} .

The SBH+disk model (Kotlařík and Kofroň, 2022):



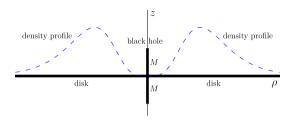
- a family of thin disks; potential first considered by (Vogt and Letelier, 2009)
- both metric functions ν , λ found analytically in **closed form** (polynomials and square roots)

Newtonian disk density
$$\propto \frac{\mathcal{M}b^{2m+1}\rho^{2n}}{(\rho^2+b^2)^{m+n+3/2}}, \quad m, n \in \mathbb{N}_0,$$
 (11)

where for every m, n, there are 2 free parameters:

- \bullet \mathcal{M} is the total mass of the disk,
- b has dimension of length.

The SBH+disk model (Kotlařík and Kofroň, 2022):



- a family of thin disks; potential first considered by (Vogt and Letelier, 2009)
- both metric functions ν , λ found analytically in **closed form** (polynomials and square roots)

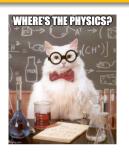
Newtonian disk density
$$\propto \frac{\mathcal{M}b^{2m+1}\rho^{2n}}{(\rho^2+b^2)^{m+n+3/2}}, \quad m,n\in \mathbb{N}_0,$$
 (11)

where for every m, n, there are **2 free parameters**:

- \bullet \mathcal{M} is the total mass of the disk,
- b has dimension of length.

Two simple physical interpretations:

- lacktriangle ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
- ② two identical counter-orbiting dust streams with surface densities $(\sigma_+ = \sigma_- \equiv \sigma/2)$ following circular geodesics

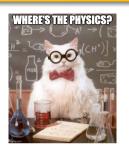


Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \frac{\nu_{,z}(z=0^+)}{2\pi}e^{\nu-\lambda}, \qquad P = \frac{\nu_{,z}(z=0^+)}{2\pi}\rho\nu_{,\rho}e^{\nu-\lambda}.$$

Two simple physical interpretations:

- ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
- 2 two identical counter-orbiting dust streams with surface densities ($\sigma_+ = \sigma_- \equiv \sigma/2$) following circular geodesics

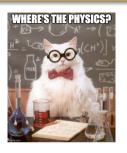


Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \frac{\nu_{,z}(z=0^+)}{2\pi} e^{\nu-\lambda}, \qquad P = \frac{\nu_{,z}(z=0^+)}{2\pi} \rho \nu_{,\rho} e^{\nu-\lambda}.$$

Two simple physical interpretations:

- ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
- ② two identical counter-orbiting dust streams with surface densities $(\sigma_+ = \sigma_- \equiv \sigma/2)$ following circular geodesics

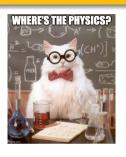


Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = rac{
u_{,z}(z=0^+)}{2\pi}e^{
u-\lambda}\,, \qquad P = rac{
u_{,z}(z=0^+)}{2\pi}
ho
u_{,
ho}e^{
u-\lambda}\,.$$

Two simple physical interpretations:

- ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
- ② two identical counter-orbiting dust streams with surface densities ($\sigma_+ = \sigma_- \equiv \sigma/2$) following circular geodesics

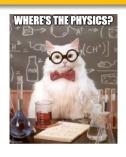


Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \underbrace{\frac{\nu_{,z}(z=0^+)}{2\pi}}_{\text{Newtonian disk density}} e^{\nu-\lambda} \,, \qquad P = \underbrace{\frac{\nu_{,z}(z=0^+)}{2\pi}}_{\text{Newtonian disk density}} \rho\nu_{,\rho} e^{\nu-\lambda} \,.$$

Two simple physical interpretations:

- lacktriangle ideal fluid with surface density σ and azimuthal pressure P (set of solid hoops)
- ② two identical counter-orbiting dust streams with surface densities ($\sigma_+ = \sigma_- \equiv \sigma/2$) following circular geodesics



Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \underbrace{\frac{\nu_{,z}(z=0^+)}{2\pi}}_{\text{Newtonian disk density}} e^{\nu-\lambda} \,, \qquad P = \underbrace{\frac{\nu_{,z}(z=0^+)}{2\pi}}_{\text{Newtonian disk density}} \rho\nu_{,\rho} e^{\nu-\lambda} \,.$$

Table of Contents

1 The Schwarzschild + disk model (SBH+disk)

2 QNMs of a scalar field

Particular results

A simpler problem: perturbations of a **massless scalar field**, which are governed by the Klein-Gordon equation

$$\Box \psi = 0 \ . \tag{12}$$

The wave equation is separable on the Schwarzschild background in the Schwarzschild coordinates (r, θ) : $\rho = \sqrt{r(r-2M)}\sin\theta$, $z = (r-M)\cos\theta$.

In particular, the ansatz

$$\psi = \frac{\psi_{\omega \ell m_z}(r)}{r} Y_{\ell m_z}(\theta, \phi) e^{-i\omega t}, \qquad (13)$$

$$\frac{\mathrm{d}^2 \Psi_{\omega \ell m_z}}{\mathrm{d}r_z^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (14)$$



A simpler problem: perturbations of a **massless scalar field**, which are governed by the Klein-Gordon equation

$$\Box \psi = 0 \ . \tag{12}$$

The wave equation is separable on the Schwarzschild background in the Schwarzschild coordinates (r, θ) : $\rho = \sqrt{r(r-2M)} \sin \theta$, $z = (r-M) \cos \theta$.

In particular, the ansatz

$$\psi = \frac{\psi_{\omega\ell m_z}(r)}{r} Y_{\ell m_z}(\theta, \phi) e^{-i\omega t}, \qquad (13)$$

$$\frac{\mathrm{d}^2 \Psi_{\omega \ell m_z}}{\mathrm{d}r_z^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (14)$$



A simpler problem: perturbations of a **massless scalar field**, which are governed by the Klein-Gordon equation

$$\Box \psi = 0 \ . \tag{12}$$

The wave equation is separable on the Schwarzschild background in the Schwarzschild coordinates (r, θ) : $\rho = \sqrt{r(r-2M)} \sin \theta$, $z = (r-M) \cos \theta$.

In particular, the ansatz

$$\psi = \frac{\psi_{\omega \ell m_z}(r)}{r} Y_{\ell m_z}(\theta, \phi) e^{-i\omega t}, \qquad (13)$$

$$\frac{d^2 \Psi_{\omega \ell m_z}}{dr_c^2} + \left[\omega^2 - V_{\text{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (14)$$



A simpler problem: perturbations of a **massless scalar field**, which are governed by the Klein-Gordon equation

$$\Box \psi = 0 \ . \tag{12}$$

The wave equation is separable on the Schwarzschild background in the Schwarzschild coordinates (r, θ) : $\rho = \sqrt{r(r-2M)} \sin \theta$, $z = (r-M) \cos \theta$.

In particular, the ansatz

$$\psi = \frac{\psi_{\omega\ell m_z}(r)}{r} Y_{\ell m_z}(\theta, \phi) e^{-i\omega t}, \qquad (13)$$

$$\frac{d^2 \Psi_{\omega \ell m_z}}{dr_{\omega}^2} + \left[\omega^2 - V_{\text{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (14)$$



However, the presence of the (axially symmetric) disk breaks this convenient property.

Cano, Fransen, and Hertog (2020): "almost separable" systems, projection method.

Chen, Chiang, and Tsao (2022): Schwarzschild black hole perturbed by a small axially symmetric deformation.

If $\epsilon \equiv \mathcal{M}/M \ll 1$, the wave equation again leads to the Reggee–Wheeler form

$$\frac{\mathrm{d}^2 \Psi_{\omega \ell m_z}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (15)$$

with the appropriately modified tortoise coordinate r_* , and

$$V_{\rm eff}(r) = V_{\rm eff}^{\rm Sch}(r) + \epsilon V_{\rm eff}^{\rm corr}(r). \tag{16}$$

However, the presence of the (axially symmetric) disk breaks this convenient property.

Cano, Fransen, and Hertog (2020): "almost separable" systems, projection method.

Chen, Chiang, and Tsao (2022): Schwarzschild black hole perturbed by a small axially symmetric deformation.

If $\epsilon \equiv \mathcal{M}/M \ll 1$, the wave equation again leads to the Reggee–Wheeler form

$$\frac{\mathrm{d}^2 \Psi_{\omega \ell m_z}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\mathrm{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (15)$$

with the appropriately modified tortoise coordinate r_* , and

$$V_{\rm eff}(r) = V_{\rm eff}^{\rm Sch}(r) + \epsilon V_{\rm eff}^{\rm corr}(r). \tag{16}$$

However, the presence of the (axially symmetric) disk breaks this convenient property.

Cano, Fransen, and Hertog (2020): "almost separable" systems, projection method.

Chen, Chiang, and Tsao (2022): Schwarzschild black hole perturbed by a small axially symmetric deformation.

If $\epsilon \equiv \mathcal{M}/M \ll 1$, the wave equation again leads to the Reggee–Wheeler form

$$\frac{d^2 \Psi_{\omega \ell m_z}}{dr_*^2} + \left[\omega^2 - V_{\text{eff}}(r)\right] \Psi_{\omega \ell m_z} = 0, \qquad (15)$$

with the appropriately modified tortoise coordinate r_* , and

$$V_{\rm eff}(r) = V_{\rm eff}^{\rm Sch}(r) + \epsilon V_{\rm eff}^{\rm corr}(r). \tag{16}$$

Table of Contents

1 The Schwarzschild + disk model (SBH+disk)

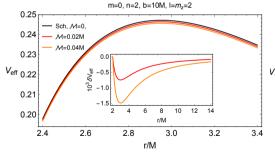
2 QNMs of a scalar field

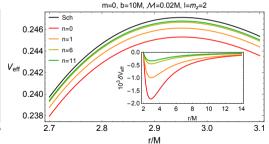
3 Particular results

Effective potential

The disk is described by **4 parameters**: 2 positive real \mathcal{M} , b, and a pair (m, n).

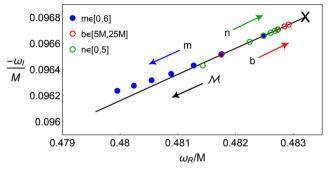
Newtonian disk density
$$\propto \frac{\mathcal{M}b^{2m+1}\rho^{2n}}{(\rho^2+b^2)^{m+n+3/2}}$$
 (17)





Universal behaviour?

QNM frequencies of the fundamental mode for $\ell=m_z=2$:



Newtonian disk density

$$\propto rac{\mathcal{M} b^{2m+1}
ho^{2n}}{(
ho^2 + b^2)^{m+n+3/2}}\,.$$

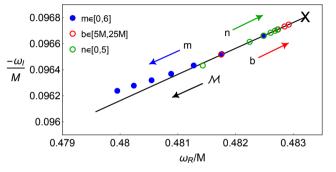
Remarks

- BH immersed in spherically symmetric (dark)matter exhibits similar behavior (Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025; Chakraborty, Compère, and Machet, 2025)
- a typical effect of the astrophysical environment?



Universal behaviour?

QNM frequencies of the fundamental mode for $\ell=m_z=2$:



Newtonian disk density

$$\propto rac{\mathcal{M} b^{2m+1}
ho^{2n}}{(
ho^2 + b^2)^{m+n+3/2}}\,.$$

Remarks:

- BH immersed in spherically symmetric (dark)matter exhibits similar behavior (Cardoso et al., 2022; Konoplya, 2021; Pezzella et al., 2025; Chakraborty, Compère, and Machet, 2025)
- a typical effect of the astrophysical environment?



Future work:

- Extend the analysis to stationary systems and to a massive scalar field

 impact on superradiant instabilities?
- Analyze gravitational QNMs and address some of the open questions → two types of perturbations involved:
 - deformation of spacetime due to the surrounding matter
 - ringdown gravitational waves
- Systematically compare with predictions of alternative theories of gravity: can the astrophysical environment imitate or obscure potential signatures of modifications to GR?

Future work:

- Extend the analysis to stationary systems and to a massive scalar field

 impact on superradiant instabilities?
- Analyze gravitational QNMs and address some of the open questions → two types of perturbations involved:
 - deformation of spacetime due to the surrounding matter
 - ringdown gravitational waves
- Systematically compare with predictions of alternative theories of gravity: can the astrophysical environment imitate or obscure potential signatures of modifications to GR?

Future work:

- Extend the analysis to stationary systems and to a massive scalar field

 impact on superradiant instabilities?
- Analyze gravitational QNMs and address some of the open questions → two types of perturbations involved:
 - deformation of spacetime due to the surrounding matter
 - ringdown gravitational waves
- Systematically compare with predictions of alternative theories of gravity: can the astrophysical environment imitate or obscure potential signatures of modifications to GR?

Brief summary

- QNMs of a scalar field propagating in the SBH+disk background
- disk flattens the effective potential
- QNMs seem to follow a universal relation
- can we disentangle environmental effects from those induced by alternative theories of gravity?

Thank you for your attention.

Questions?

Brief summary

- QNMs of a scalar field propagating in the SBH+disk background
- disk flattens the effective potential
- QNMs seem to follow a universal relation
- can we disentangle environmental effects from those induced by alternative theories of gravity?

Thank you for your attention.

Questions?



References I

- Barausse, Enrico, Vitor Cardoso, and Paolo Pani (2014). "Can Environmental Effects Spoil Precision Gravitational-Wave Astrophysics?" *Physical Review D* 89(10), p. 104059. arXiv: 1404.7149 [astro-ph, physics:gr-qc].
- Cheung, Mark Ho-Yeuk, Kyriakos Destounis, Rodrigo Panosso Macedo,
 Emanuele Berti, and Vitor Cardoso (2022). "Destabilizing the Fundamental Mode of
 Black Holes: The Elephant and the Flea". Physical Review Letters 128(11),
 p. 111103. arXiv: 2111.05415 [astro-ph, physics:gr-qc, physics:hep-th].
- Cardoso, Vitor, Kyriakos Destounis, Francisco Duque, Rodrigo Panosso Macedo, and Andrea Maselli (2022). "Black Holes in Galaxies: Environmental Impact on Gravitational-Wave Generation and Propagation". *Physical Review D* 105(6), p. L061501.
- Konoplya, R. A. (2021). "Black Holes in Galactic Centers: Quasinormal Ringing, Grey-Body Factors and Unruh Temperature". *Physics Letters B* 823, p. 136734.

References II

- Pezzella, Laura, Kyriakos Destounis, Andrea Maselli, and Vitor Cardoso (2025). "Quasinormal Modes of Black Holes Embedded in Halos of Matter". *Physical Review D* 111(6), p. 064026. arXiv: 2412.18651 [gr-qc].
- Berti, Emanuele, Vitor Cardoso, Mark Ho-Yeuk Cheung, Francesco Di Filippo, Francisco Duque, Paul Martens, and Shinji Mukohyama (2022). "Stability of the Fundamental Quasinormal Mode in Time-Domain Observations against Small Perturbations". *Physical Review D* 106(8), p. 084011. arXiv: 2205.08547 [astro-ph, physics:gr-qc, physics:hep-th].
- Cardoso, Vitor, Shilpa Kastha, and Rodrigo Panosso Macedo (2024). "Physical Significance of the Black Hole Quasinormal Mode Spectra Instability". *Physical Review D* 110(2), p. 024016.
- James, Oliver, Eugénie von Tunzelmann, Paul Franklin, and Kip S. Thorne (2015). "Gravitational Lensing by Spinning Black Holes in Astrophysics, and in the Movie Interstellar". Classical and Quantum Gravity 32(6), p. 065001.

References III

- Kotlařík, Petr and David Kofroň (2022). "Black Hole Encircled by a Thin Disk: Fully Relativistic Solution*". *The Astrophysical Journal* 941(1), p. 25.
- Vogt, D. and P. S. Letelier (2009). "Analytical Potential-Density Pairs for Flat Rings and Toroidal Structures". *Monthly Notices of the Royal Astronomical Society* 396(3), pp. 1487–1498.
- Cano, Pablo A., Kwinten Fransen, and Thomas Hertog (2020). "Ringing of Rotating Black Holes in Higher-Derivative Gravity". *Physical Review D* 102(4), p. 044047.
- Chen, Che-Yu, Hsu-Wen Chiang, and Jie-Shiun Tsao (2022). "Eikonal Quasinormal Modes and Photon Orbits of Deformed Schwarzschild Black Holes". *Physical Review D* 106(4), p. 044068. arXiv: 2205.02433 [astro-ph, physics:gr-qc, physics:hep-th].
- Chakraborty, Sumanta, Geoffrey Compère, and Ludovico Machet (2025). *Tidal Love Numbers and Quasi-Normal Modes of the Schwarzschild-Hernquist Black Hole*. arXiv: 2412.14831 [gr-qc].