Self-gravitating spinning mesonic tubes

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Motivation

- Aichelburg-Sexl metric describes the gravitational field of an ultrarelativistic spinless point particle.
- Gyraton metric represents a beam of spinning null fluid
- Gyraton solutions are usually sourced by an effective stress energy tensor singular on the propagation axis
- Usually no fundamental action principle is used
- Infinite extension in space and time

- GOAL OF THIS TALK:

- Show that there exist exact self gravitating gyraton solutions with no sharp boundaries and free of curvature or mater singularities. Moreover these solutions are PULSED

  Case of gauged non linear Einstein-non-linear Sigma model

  (Exist also in the case of gauged Einstein-Skyrme model with cosmological constant but time constraint in this talk)

The action principle

- Einstein- gauged Skyrme model

$$I[g,U,A]=\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa}+\frac{K}{4}\mathrm{Tr}[L^{\mu}L_{\mu}]+\frac{K\lambda}{32}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\ , \tag{1}$$

$$L_{\mu}=U^{-1}D_{\mu}U=L^{a}_{\mu}t_{a}\ ,\quad G_{\mu\nu}=[L_{\mu},L_{\nu}]\ ,$$

where $U(x)\in SU(2)$, g is the metric determinant, R is the Ricci scalar, Λ is the cosmological constant, K and λ are experimentally fixed positive coupling constants (in the special case that λ is zero the matter action is also known as Non-Linear-Sigma-Model), and $t_a=i\sigma_a$ are the generators of the $SU(2)$ Lie group, being σ_a the Pauli matrices. In Eq. (1), D_μ denotes the covariant derivative associated to the $U(1)$ gauge field A_μ, that acts as

$$D_{\mu}U=\nabla_{\mu}U+A_{\mu}U\hat{O}\ ,\quad \hat{O}=U^{-1}\left[t_{3},U\right]\ , \tag{2}$$

Matter field and Metric ansatz

- Ansatz

$$U = e^{F_1(x^\mu)\cdot t_3} e^{F_2(x^\mu)\cdot t_2} e^{F_3(x^\mu)\cdot t_3} \ ,$$

where the three degrees of freedom of the $SU(2)$ group element are

$$F_1(x^\mu) = \frac{p}{2}\varphi \ , \qquad F_2(x^\mu) = H(r) \ , \qquad F_3(x^\mu) = G(u) \ .$$

A metric Ansatz in the Kundt class that leads to a compatible equations system (as we will see below) is the following

$$ds^2 = -f du^2 + e^{R_1}(dr^2 + r^2 d\varphi^2) + 2dudv + 2R_2 dud\varphi \ , \qquad x^\mu = \{u, v, r, \varphi\} \ , \tag{9}$$

with $f = f(u,r)$, $R_1 = R_1(r)$ and $R_2 = R_2(u,r)$. Here u and v are null coordinates; $u = t - z$ and $v = t + z$. The range of the spatial coordinates $\{r, \varphi, z\}$ are

$$0 < r < \infty \ , \quad 0 \le \varphi < 2\pi \ , \quad -\infty < z < \infty \ .$$

On the other hand, for the Maxwell potential we assume

$$A_\mu = (B, 0, 0 - \frac{p}{4}) \ . \tag{10}$$

where $B = B(u,r)$.

Generic field equations

- Eliminating the G(u) dependence for eq. of motion by redefining:

$$B(u,r) = \frac{1}{2}\dot{G}(u)(1 - P(r)) \ ,$$
$$R_2(u,r) = \dot{G}(u)V(r) \ ,$$
$$f(u,r) = (\dot{G}(u))^2 M(r) \ ,$$

The 3 Skyrme equations reduce to one equation and inserting Einstein field equations reads:

$$\lambda(2 - p\sin(H) + rR_1') = 0 \ .$$

- Einstein field equations:

$$(H')^2 - \frac{p^2}{4r^2}\cos^2(H) = 0\ , \tag{17}$$

$$K\kappa\lambda p^2\cos^2(H)\,(H')^2 - 8\Lambda r^2 e^{2R_1} = 0\ , \tag{18}$$

$$R_1'' + \frac{1}{r}R_1' + \frac{K\kappa p^2}{2r^2}\cos^2(H) + \frac{K\kappa\lambda p^4}{16r^4}\cos^4(H)e^{-R_1} + 2\Lambda e^{R_1} = 0\ , \tag{19}$$

$$V'' - \left(R_1' + \frac{1}{r}\right)V' + K\kappa\, p\, e^{R_1}\cos^2(H) + \frac{K\kappa\lambda p^3}{4r^2}\cos^4(H) = 0\ , \tag{20}$$

$$M'' + \frac{1}{r}M' + \frac{e^{-R_1}}{r^2}\left(V'\right)^2 - \frac{\kappa}{2}\left(P'\right)^2 - 2K\kappa e^{R_1}\left(P^2\sin^2(H) + \cos^2(H)\right) - \frac{K\kappa\lambda p^2}{4r^2}\left[P^2\sin^2(H) + 2\cos^4(H)\right] = 0\ . \tag{21}$$

- The maxwell equation reads
- $$P'' + \frac{1}{r}P' - 4KP\sin^2 H\left[e^{R_1} + \lambda\left(H'^2 + \frac{p^2}{4r^2}\cos^2 H\right)\right] = 0\ .$$
- As nothing depends on G(u) the function can be taken with compact support.
- The function H(r) can be immediately integrated

$$H(r) = \arcsin\left(\frac{c_1 r^p - 1}{c_1 r^p + 1}\right)\ .$$

Eq. 18) leads to two branches:

- a) set $\Lambda = \lambda = 0$
- b) solve for R_1 in terms of H(r)

Case Λ=λ=0

- In this case the e.o.m. reduce to

$$P'' + \frac{1}{r}P' - 4KPe^{R_1}\sin^2(H) = 0\ , \tag{24}$$

$$R_1'' + \frac{1}{r}R_1' + \frac{K\kappa p^2}{2r^2}\cos^2(H) = 0\ , \tag{25}$$

$$V'' - \left(\frac{1}{r} + R_1'\right)V' + K\kappa p e^{R_1}\cos^2(H) = 0\ , \tag{26}$$

$$M'' + \frac{1}{r}M' - \frac{\kappa}{2}(P')^2 - K\kappa e^{R_1}(1+\cos(2H) + 2P^2\sin^2(H)) + \frac{e^{-R_1}}{r^2}(V')^2 = 0\ , \tag{27}$$

- eq. 25) can be integrated
- $R_1(r) = (2K\kappa p + c_3)\log(r) - 2K\kappa\log(1 + c_1 r^p) + c_2\ .$
- The integration constant c_2 can be elimated by a suitable rescaling of r and the absence of conical singularities leads to the condition
- $$c_3 = -2K\kappa|p|\ ,$$

Avoiding conical singularity

$$X = \frac{2}{m} r^{m/2} \; , \qquad \text{where} \qquad m = 2(K\kappa|p| + 1) + c_3 \; .$$

Then, the induced metric ds_2^2 on the 2-surface with constant u and v approximates to

$$ds_2^2 \approx dX^2 + \frac{m^2}{4} X^2 d\varphi^2 \; ,$$

near the symmetry axis. To remove the conical singularity along the symmetry axis, we should impose $m = 2$, that means

$$c_3 = -2K\kappa|p| \; , \tag{29}$$

$$g_{rr} = e^{R_1} = \frac{1}{\left(1 + c_1 r^{|p|}\right)^{2K\kappa}} \ ,$$

- Also eq. 26) can be integrated in terms of hypergeometric functions

$$V(r) = -2K\kappa r^2 {}_2F_1(1 + 2K\kappa, 2/|p|, 1 + 2/|p|; -c_1 r^{|p|}) \\ + V_1 r^{2+p} {}_2F_1(2K\kappa, 2/|p|, 1 + 2/|p|; -c_1 r^{|p|}) \ .$$

- Regularity in r=0 forces p>2
- Now are still missing the functions P and M however the curvature invariants do not depend on P and M so we can check the regularity.

- The Ricci scalar takes the form

$$R = -\frac{e^{-R_1}}{r}(R_1' + rR_1'') = \frac{2c_1 K\kappa p^2 r^{p-2}}{(1 + c_1 r^p)^{2(1-K\kappa)}}$$

- As p>2 and Kκ <<1 it is regular in zero and at infinity.Its maximum lies on a tube around the propagation axis. All other curvature invariants are zero or powers of the Ricci scalar.

- Exact solution:

  we can chose P=0 (pure gauge Maxwell field) then eq. 24) is satisfied and M can be integrated in this case

$$
\begin{aligned}
M(r) =&4K\kappa r^2\Big\{ \frac{2c_1 r^p}{(2+p)^2}\, {}_3F_2\Big(\{1+2/p, 1+2/p, 2(1+K\kappa)\}, \{2(1+1/p), 2(1+1/p)\}; -c_1 r^p\Big) \\
&- K\kappa\, {}_3F_2\Big(\{2/p, 2/p, 2(1+K\kappa)\}, \{1+2/p, 1+2/p\}; -c_1 r^p\Big)\Big\} + M_0 \ ,
\end{aligned}
$$

- The components of the stress energy tensor read

$$
\begin{aligned}
T_{uu} =& \frac{e^{-R_1}\dot{G}^2}{8r^2}\left(KM(r)\left(4r^2H'^2+p^2\cos^2(H)\right)+2r^2\left(4Ke^{R_1}\left(P^2\sin^2(H)+\cos^2(H)\right)+P'^2\right)\right)\\
T_{uv} =& -\frac{Ke^{-R_1}}{8r^2}\left(4r^2H'^2+p^2\cos^2(H)\right)\\
T_{u\varphi} =& \frac{K\dot{G}}{8r^2}\left(4pr^2\cos^2(H)-e^{-R_1}V\left(4r^2H'^2+p^2\cos^2(H)\right)\right)\\
T_{rr} =& \frac{K}{8r^2}\left(4r^2H'^2-p^2\cos^2(H)\right)=-\frac{1}{r^2}T_{\varphi\varphi}
\end{aligned}
$$

- Some components do not correspond to a null fluid but with our regularity conditions asymptotically the the non null fluid components of the energy stress tensor go faster to zero than the null fluid components

- The r-r and φ-φ are zero because of the equations of motion

$$T_{uu} \sim \frac{1}{r^p} \quad ; \quad T_{u\varphi} \sim \frac{1}{r^p} \quad ; \quad T_{uv} \sim \frac{1}{r^{p+2+2K\kappa}}$$

- As asymptotically the metric can not be distinguished from a gyraton the angular momentum per unit of length can be calculated with standard methods (see Podolsy, Steinbauer,Svarc; Phys. Rev. D90 (2014) 4, 044050). The total angular momentum is finite as G(u) has compact support.

$$J(u) = -\frac{1}{8\pi} \lim_{r\to\infty} \oint R_2 \, d\phi = -\frac{\pi K \kappa}{p^2 C_1^{2/|p|}} \csc\left(\frac{2\pi}{|p|}\right) \dot{G} \ .$$

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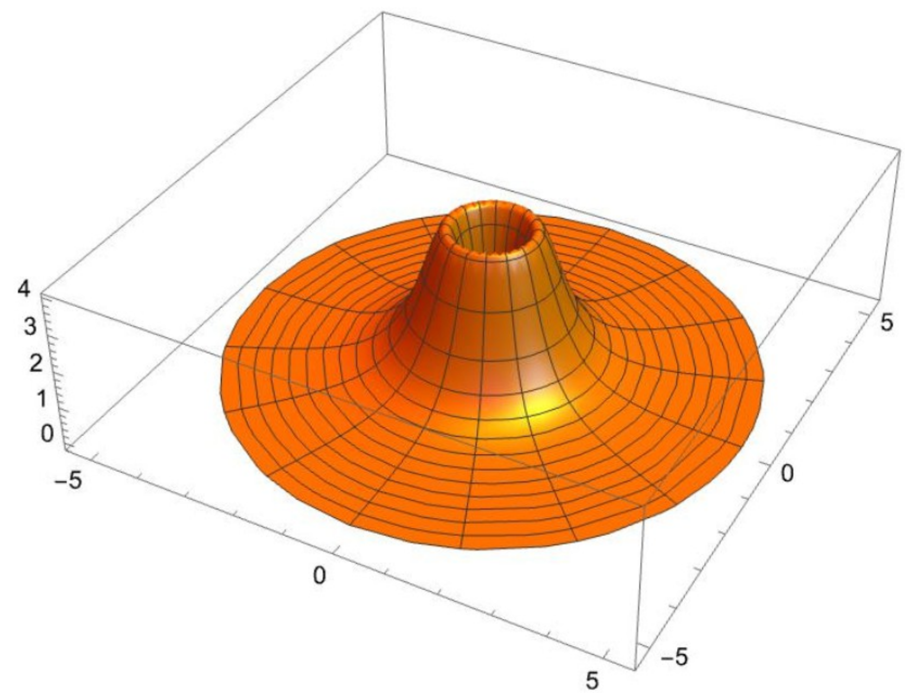

FIG. 1: Energy density for the exact asymptotically flat self-gravitating spinning tube, in the $r - \varphi$ plane . We have used the following values: $K = c_1 = 1$, $p = 3$, $\kappa = 1/5$, $m_0 = 0$ and $\dot{G} = 1$.

also the energy density has finite extension in time and propagation axis: it describes a pulse.

Topological charge density

$$\rho_B = -\frac{3p}{2}\frac{d}{dr}\left(P(r)\cos^2 H(r)\right) .$$

- If we decide NOT to set P=0 it can be integrated as a Heun function and M satisfies then a linear ODE.