Dynamics of Brane-world models

Ifigeneia Klaoudatou
(The American College of Greece, Deree)

Joint work with

Prof. Ignatios Antoniadis
(LPTHE, CNRS/Sorbonne University, HEP Research Unit, Faculty of Science, Chulalongkorn University)
&
Prof. Spiros Cotsakis
(Clare Hall, University of Cambridge, Institute of Gravitation and Cosmology, RUDN University)

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Brane-worlds

- The observed universe is modeled by a 4d hypersurface situated at a fixed position of an extra spatial dimension.
- The whole spacetime is five dimensional: there are four dimensions of space, out of which only three are spanned by the hypersurface and one dimension of time. The hypersurface is called a **3-brane** while the full higher dimensional space is called the **bulk**.

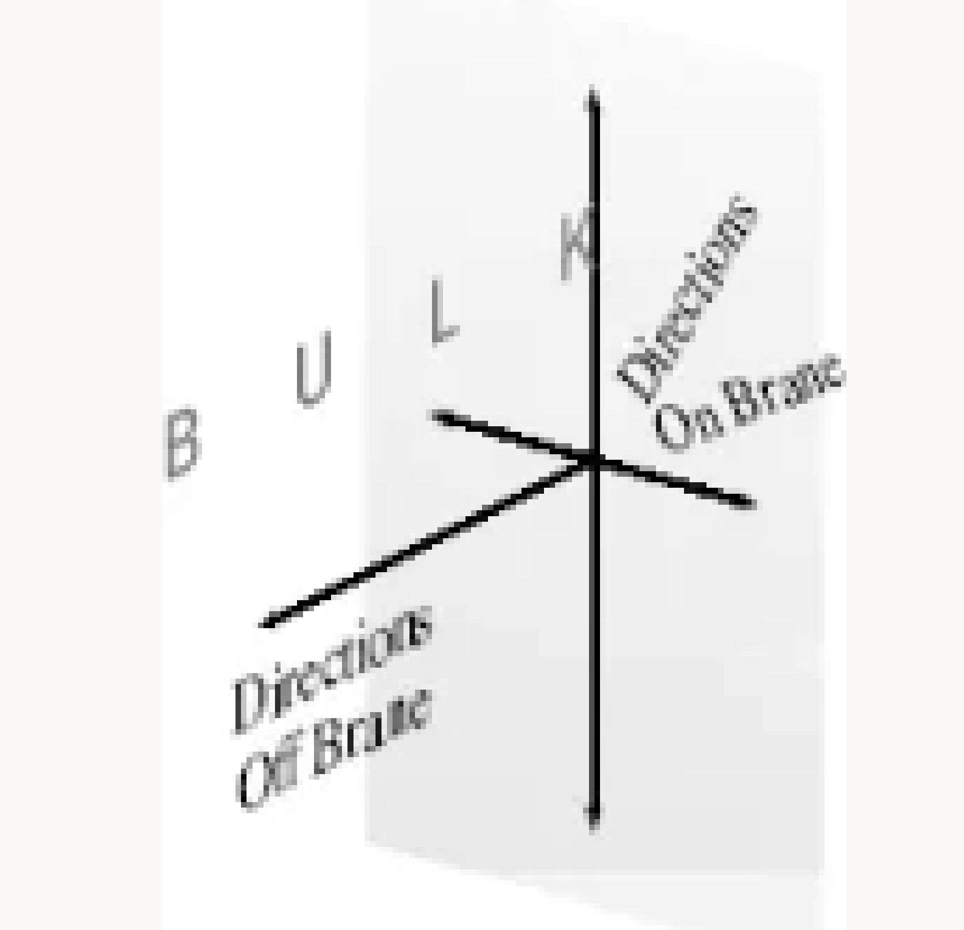

Figure from *Warped Passages*, Lisa Randall

Brane-worlds

- A brane can trap particles and forces making it impossible for them to escape.
- However, it does interact with the matter and forces in the bulk. This happens because gravity naturally extends over all dimensions of space and time, and it can therefore influence the fields that are constrained on the brane.
- Other bulk matter fields can also interact with the fields on the brane, and the strength of this interaction is controlled by a coupling function that is model-dependent.

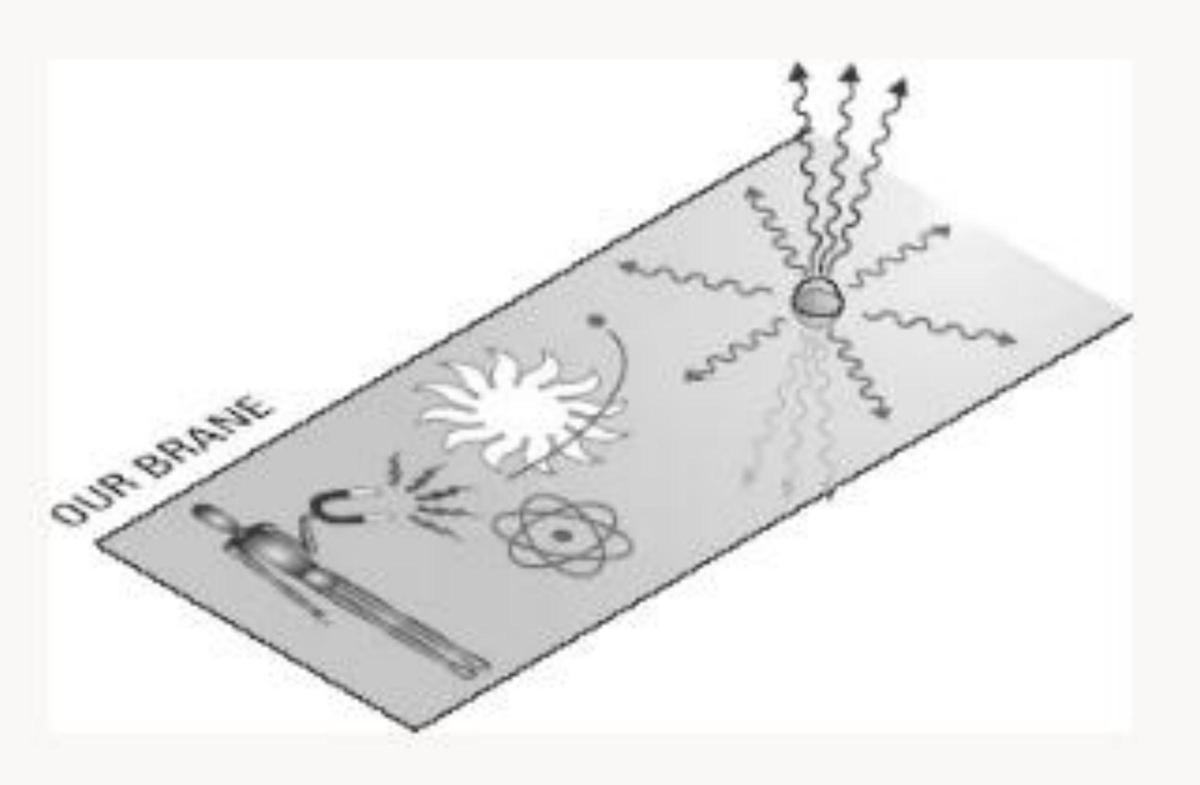

Figure from *Warped Passages*, Lisa Randall

Brane-worlds

- Higher-dimensional models propose alternative approaches towards understanding and hopefully improving our view on challenging issues in cosmology and particle physics.
- We focus on a particular class of brane-worlds because it offers an interesting implication on the cosmological constant problem (cc-problem).

The cosmological constant problem

- The cc-problem arises from the disagreement between predictions from quantum field theories and observations regarding the value of the cosmological constant.
- The theoretical quantum corrections to the cosmological constant are naturally some 120 orders of magnitude higher than its observed value.
- Based on theory, the huge value of the cosmological constant would automatically imply a huge value of the vacuum energy, which would in turn give rise to a highly curved universe, a prediction that is not compatible with observations.

Brane-worlds: open questions

- The brane-world of *N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, R. Sundrum, A small cosmological constant from a large extra dimension,* consisted of a flat 3-brane embedded in a 5d bulk filled with a massless scalar field and had a finite-distance singularity.
- This singularity was believed to act as a reservoir of energy through which energy leaked from the brane to the bulk, allowing for a small cosmological constant to be observed on the brane.

Brane-worlds: open questions

Open questions

- Is the finite-distance singularity a generic feature of these models?
- Under what conditions can it be avoided?
- Can the problem of the cosmological constant be resolved in the framework of these more general models?

Brane-worlds: open questions

We generalized the model of *N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, R. Sundrum* by:

- Substituting the scalar field with an analog of perfect fluid: EOS $p = \gamma\rho$.
- Substituting the scalar field with a non-linear fluid: EOS $p = \gamma\rho^{\lambda}$.

Setup of brane-world

- A 3-brane is embedded in a 5d bulk filled with a fluid with ‘pressure’ p and ‘density’ ρ that are functions of the fifth dimension Y. The bulk metric is

$$g_5 = a^2(Y) g_4 + dY^2$$

where g_4 is the 4d flat metric.

- The energy-momentum tensor of the fluid is

$$T_{AB} = (\rho + p) u_A u_B - p g_{AB}$$

where $u_{\mathrm{A}} = (0,0,0,0,1)$ and A,B=1,2,3,4,5.

Setup of brane-world

- Field equations $G_{AB} = \kappa_5^2 T_{AB}$ become

$$6\frac{a'^2}{a^2} = \kappa_5^2 \rho$$

$$\frac{a''}{a} = -\frac{\kappa_5^2}{6}(\rho + 2p)$$

- The energy-momentum conservation gives

$$\rho' + 4\frac{a'}{a}(\rho + p) = 0$$

Linear bulk fluid with EOS $p = \gamma\rho$

Linear bulk fluid with EOS $p = \gamma\rho$

- For a linear fluid $p = \gamma\rho$, the continuity equation gives gives:

$$\rho = c_1 a^{-4(\gamma+1)}$$

and substitution to the first field equation gives

$$a = \left(2(\gamma+1)\left(\pm\sqrt{\frac{\kappa_5^2 c_1}{6}} Y + c_2 \right) \right)^{1/(2(\gamma+1))}$$

also

$$\rho = c_1 \left(2(\gamma+1)\left(\pm\sqrt{\frac{\kappa_5^2 c_1}{6}} Y + c_2 \right) \right)^{-2}$$

Linear bulk fluid with EOS $p = \gamma\rho$

- It follows that there is a singularity at finite-distance

$$Y = Y_0 = \pm c_2 \sqrt{\frac{6}{\kappa_5^2 c_1}}$$

with

- $\gamma > -1$: $a \to 0,\ \rho \to \infty, Y \to Y_0$
- $\gamma < -1$: $a \to \infty,\ \rho \to \infty,\ Y \to Y_0$

Linear bulk fluid with EOS $p = \gamma\rho$

- To avoid the singularity, we can place the brane at the origin $Y = 0$ and make the following choices:

  - $\gamma > -1,\ c_2 \geq 0,$

$$a = \left(2(\gamma+1)\left(\sqrt{\frac{\kappa_5^2 c_1}{6}}|Y| + c_2 \right) \right)^{1/(2(\gamma+1))}$$

and

$$\rho = c_1 \left(2(\gamma+1)\left(\sqrt{\frac{\kappa_5^2 c_1}{6}}|Y| + c_2 \right) \right)^{-2}$$

Linear bulk fluid with EOS $p = \gamma\rho$

- $\gamma < -1, \;\; c_2 \le 0,$

$$a = \left(2(\gamma+1)\left(-\sqrt{\frac{\kappa_5^2 c_1}{6}}|Y| + c_2\right)\right)^{1/(2(\gamma+1))}$$

and

$$\rho = c_1\left(2(\gamma+1)\left(-\sqrt{\frac{\kappa_5^2 c_1}{6}}|Y| + c_2\right)\right)^{-2}$$

Linear bulk fluid with EOS $p = \gamma\rho$

- We impose continuity of α and ρ.
- Take into account the jump of the extrinsic curvature $K_{\alpha\beta} = 1/2(\partial g_{\alpha\beta}/\partial Y)$ $(\alpha, \beta = 1, 2, 3, 4)$:

$$K^{+}_{\alpha\beta} - K^{-}_{\alpha\beta} = -\kappa_5^2\left(S_{\alpha\beta} - \frac{1}{3}g_{\alpha\beta}S\right)$$

where $S_{\alpha\beta} = -g_{\alpha\beta}f(\rho)$, $f(\rho)$ is the brane tension and

$S = g^{\alpha\beta}S_{\alpha\beta}$.

- For $\gamma < -1$, $c_2 \leq 0$: $c_1^+ = c_1^-$ and $c_2^+ = c_2^-$ (c_i^+ is the value of c_i at $Y > 0$)

$$f(\rho) = \frac{\sqrt{6}}{2\kappa_5(\gamma + 1)}\frac{\sqrt{c_1}}{c_2}$$

Linear bulk fluid with EOS $p = \gamma\rho$

Summary:

The finite-distance singularity can be avoided by constructing a matching solution. However, the matching solution cannot at the same time satisfy the requirements of

- Energy conditions: require at least $\gamma > -1$
- Localizing gravity: requires $-2 < \gamma < -1$

Non-linear bulk fluid with EOS $p = \gamma\rho^{\lambda}$

Non-linear bulk fluid with EOS $p = \gamma\rho^\lambda$

- Inputting $p = \gamma\rho^\lambda$ in the energy conditions gives the common restriction

$$\gamma + \rho^{1-\lambda} \geq 0$$

- Taking this into account while integrating the continuity equation we find

$$\rho = \left(-\gamma + c_1 a^{4(\lambda-1)}\right)^{1/(1-\lambda)}.$$

- To avoid the singularity for

$$\lambda > 1 \qquad \text{and} \qquad a^{4(\lambda-1)} = \frac{\gamma}{c_1}$$

we take $\gamma < 0$.

Solutions for $\lambda < 1$: finite-distance singularities

- Substituting ρ in the first field equation we find:

$$\int \frac{a}{(c_1 - \gamma a^{4(1-\lambda)})^{1/(2(1-\lambda))}} da = \pm \frac{\kappa_5}{\sqrt{6}} \int dY$$

- **Solution for $\lambda < 1$:**

$$\pm Y + c_2 = \frac{\sqrt{6}}{2\kappa_5} c_1^{1/(2(\lambda-1))} a^2 \; {}_2F_1\left(\frac{1}{2(1-\lambda)}, \frac{1}{2(1-\lambda)}, \frac{1}{2(1-\lambda)} + 1; \frac{\gamma}{c_1} a^{4(1-\lambda)}\right)$$

Solutions for $\lambda < 1$: finite-distance singularities

where

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \quad |z| < 1, $$

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, $$

$$ 0 < Re(b) < Re(c). $$

Finite-distance singularity at $Y \to \pm c_2$ with $a \to 0$

Solutions for $\lambda > 1$: regularity

- The integral on the LHS can be integrated directly for

$$\lambda = \frac{2k+1}{2k}, \ k \text{ positive integer.}$$

Take $\lambda = 3/2$ the solution is

$$\pm Y + c_2 = \frac{\sqrt{6}}{\kappa_5}\left(\frac{c_1}{2}a^2 - \gamma \ln a\right)$$

then

$$a \to 0, \ \rho \to (-\gamma)^{1/(1-\lambda)}, \ p \to -(-\gamma)^{1/(1-\lambda)}, \text{ as } Y \to \pm\infty$$

$$a \to \infty, \ \rho \to 0, \ p \to 0, \text{ as } Y \to \pm\infty$$

Solutions for $\lambda > 1$: matching solution

- A matching solution is

$$|Y| = \frac{\sqrt{6}}{\kappa_5}\left(-\frac{c_1}{2}a^2 + \gamma \ln a - \frac{\gamma}{2} - \gamma \ln \sqrt{\frac{-\gamma}{c_1}}\right)$$

$$a'_+(0) - a'_-(0) = 2a'_+(0) = -\frac{\kappa_5^2}{3} f\big(\rho(0)\big)a(0)$$

$$f\big(\rho(0)\big) = \frac{\sqrt{6}}{2\kappa_5\gamma}$$

Solutions for $\lambda > 1$: λ arbitrary

For all $\lambda > 1$,

- The asymptotic behaviors are:
  - $a \to 0, \rho \to (-\gamma)^{1/(1-\lambda)}, p \to -(-\gamma)^{1/(1-\lambda)}$ as $Y \to \pm\infty$
  - $a \to \infty, \rho \to 0, p \to 0$ as $Y \to \pm\infty$
- We can construct matching solutions that successfully localize gravity on the brane.

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

- We model the non-linear bulk fluid with EOS

$$\boldsymbol{p} = -\rho^2$$

with a scalar field having

$$\rho = \frac{\phi'^2}{2} - V(\phi) \text{ and } \boldsymbol{p} = \frac{\phi'^2}{2} + V(\phi)$$

we find

$$\boldsymbol{\phi'^2 = -1 + 2V(\phi) + \sqrt{1 - 8V(\phi)}}$$

with $-1 < V(\phi) < 0$.

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

- Inputting this in the equation of motion

$$\phi'' + 4\frac{a'}{a}\phi' = \dot{V}(\phi)$$

and using the first field equation

$$6\frac{a'^2}{a^2} = \kappa_5^2\left(\frac{\phi'^2}{2} - \mathrm{V}(\phi)\right)$$

we find

$$V(\phi) = -\frac{2e^{2k(\phi+c)}\left(1 + 6e^{2k(\phi+c)} + e^{4k(\phi+c)}\right)}{\left(1 + e^{2k(\phi+c)}\right)^4}$$

where $k = \frac{\sqrt{2}}{\sqrt{3}}\kappa_5$.

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

- Substituting $V(\phi)$ in

$$\phi'^2 = -1 + 2V(\phi) + \sqrt{1 - 8V(\phi)}$$

And integrating we find

$$e^{-k(\phi+c)} + e^{k(\phi+c)} + \ln\left(\frac{e^{k(\phi+c)} - 1}{e^{k(\phi+c)} + 1}\right)^2 = \pm 2\mathrm{k}(\mathrm{Y} + \mathrm{c}_1)$$

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

- Also, we find an expression for H

$$H = \frac{a'}{a} = \pm \frac{k e^{k(\phi+c)}}{1 + e^{2k(\phi+c)}}$$

and following this we find an expression for a:

$$a = c_2 \sqrt{\left| e^{k(\phi+c)} - e^{-k(\phi+c)} \right|}$$

valid for

$$a = c_2 \sqrt{\frac{e^{k(\phi+c)}}{\left| e^{2k(\phi+c)} - 1 \right|}}$$

Scalar field realization of the non-linear bulk fluid: An example with $\gamma = -1$ and $\lambda = 2$

- The possible asymptotic behaviors of Y, V, H and a are:

	Y	V	H	a
$\phi \to -c^-$	$\pm\infty$	-1	$\pm k/2$	$0, \infty$
$\phi \to -c^+$	$\pm\infty$	-1	$\pm k/2$	$0, \infty$
$\phi \to -\infty$	$\pm\infty$	0	0	$0, \infty$
$\phi \to \infty$	$\pm\infty$	0	0	$0, \infty$

Conclusions

We have studied brane-worlds with a flat 3-brane in a 5d-bulk, filled with fluids with EOS

$$p = \gamma \rho^{\lambda}$$

and an example of a scalar field realization

- For $\lambda = 1$, the solutions are singular. A regular matching solution can be constructed but it cannot at the same time satisfy energy conditions and the requirement of a finite Planck mass on the brane.
- For $\lambda < 1$, the solutions have a finite-distance singularity.
- For $\lambda > 1$ and $\gamma < 0$, the solutions are regular, satisfy energy conditions and localize gravity on the brane.
- For λ=2 and γ=−1, it is possible to describe the non-linear fluid with regular behavior by a scalar field with a potential.

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