Robinson–Trautman spacetimes in the Einstein–Gauss–Bonnet theory

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Plan of the talk

Motivation: Why study RT spacetimes in EGB theory?

General relativity in $D = 4$: the best theory of gravity we have!

General relativity in $D > 4$: interesting toy model

RT in $D = 4$ **GR**: Weyl algebraic type II and more special
(black holes, C-metric, gravitational waves, ...)

RT in $D > 4$ **GR**: Weyl algebraic type D
(black holes only)

Is this $D = 4$ versus $D > 4$ discrepancy due to:
the gravity theory or the geometric nature of RT class?

Study RT class in the natural $D > 4$ GR extension, i.e., **in EGB theory**!

Preliminaries: General Relativity

General Relativity formulated by a variational principle:

- the vacuum Einstein–Hilbert action in any dimension D

$$S = \int d^D x \sqrt{-g} \frac{1}{\mathsf{k}} \Big(R - 2\Lambda \Big)$$

- $\delta S = 0$: the equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

Einstein's equations: system of 2nd order non-linear PDEs

Preliminaries: Extensions of GR

Why to modify Einstein's general relativity?

- cosmological issues: dark energy, dark matter
- small scales: singularities, compatibility with quantum description

How to modify Einstein's general relativity?

- *number of dimensions*
- various (exotic) matter contributions
- *geometry*

Preliminaries: EGB theory – action

- non-trivial representative of Lovelock gravities (Lovelock 1971)
- the heterotic string theory limit for low energies e.g. (Gross and Sloan 1987)

It is introduced via the **Gauss–Bonnet term** in the action:

$$S = \int d^D x \sqrt{-g} \left[\frac{1}{k}(R - 2\Lambda) + \gamma L_{GB} \right]$$

where the L_{GB} stands for

$$L_{GB} = R^{cdef} R_{cdef} - 4R^{cd} R_{cd} + R^2$$

where R is the Ricci scalar, and Λ, k and γ are constants ($\gamma = 0 \Leftrightarrow$ GR).

EGB theory preserves the 2nd order field equations and is thus the natural extension of general relativity to higher dimensions.

Preliminaries: EGB theory – field equations

The field equations are obtained using $\delta S = 0$:

$$R_{ab} - \tfrac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} + 2k\gamma\, H_{ab} = 0$$

where

$$H_{ab} \equiv R\, R_{ab} - 2R_{acbd}\, R^{cd} + R_{acde}\, R_b{}^{cde} - 2R_{ac}\, R_b{}^{c} - \tfrac{1}{4}\, g_{ab}\, L_{GB}$$

with

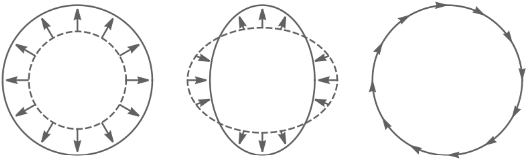
$$L_{GB} \equiv R_{cdef}\, R^{cdef} - 4\, R_{cd}\, R^{cd} + R^2$$

Our goal is:

- to explicitly derive and analyse these 2nd order equations for the Robinson–Trautman geometric ansatz
- to compare properties of obtained solutions in the EGB gravity with those of $D > 4$ GR

Preliminaries: Geometry of null congruences

The transverse behavior of a geodesic congruence generated by a null vector field ***k*** is characterized by **optical scalars**.



Expansion:

$$\Theta = \frac{1}{D-2} k^a{}_{;a}$$

Shear:

$$\sigma^2 = k_{(a;b)} k^{a;b} - \frac{1}{D-2} \left(k^a{}_{;a}\right)^2$$

Twist:

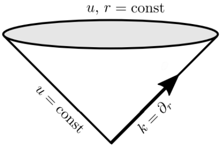
$$A^2 = -k_{[a;b]} k^{a;b}$$

Robinson–Trautman geometries: spacetimes admitting non-twisting, shear-free and *expanding* null geodesic congruence.

Preliminaries: RT geometries – adapted coordinates

Twist-free condition $k_{[a;b]} = 0 \Leftrightarrow \exists$ **null foliation** with $\boldsymbol{k}$ normal

D-dim non-twisting spacetime in $r,\ u,\ x^p$ $(p = 2, .., D-1)$ coordinates:



- $u = \text{const}$ labels null hypersurfaces
- $\boldsymbol{k} = \partial_r$ is a generator of a non-twisting congruence
- r is an affine parameter along such a congruence
- $u = \text{cst.} \wedge r = \text{cst.}$ is $D-2$-dim space g_{pq}

The non-twisting metric g_{ab} becomes:

$$\mathrm{d}s^2 = g_{pq}(r,u,x)\,\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\,\mathrm{d}x^p\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2$$

Shear-free condition:

$$g_{pq,r} = 2\Theta g_{pq} \quad \Leftrightarrow \quad g_{pq} = \exp\left(2\int \Theta(r,u,x)\,\mathrm{d}r\right) h_{pq}(u,x)$$

Results: *rr*-component

For *simplicity* and in *analogy with GR* we set:

$$g_{up}(r,u,x) = 0$$

and the RT metric, we want to study, becomes:

$$ds^2 = g_{uu}(r,u,x)du^2 - 2dudr + \mathcal{R}(r,u,x)h_{pq}(u,x)dx^p dx^q$$

The EGB field equations rr-**component** takes the form:

$$(\Theta_{,r} + \Theta^2)\big[D - 2 + 2\kappa\gamma(D-4)\big({}^sR + (D-2)(D-3)\Theta^2 g_{uu} - (D-3)\Theta g^{mn} g_{mn,u} + 4(D-3)g^{mn}\Theta_{,m}\Theta_{,n}\big)\big] = 0$$

Two branches of solutions w.r.t. $\Theta_{,r} + \Theta^2$

- $\Theta_{,r} + \Theta^2 = 0 \;\rightarrow\; \Theta \approx r^{-1}$ ... GR-like behavior
- $\Theta_{,r} + \Theta^2 \neq 0 \;\rightarrow\;$ e.g. constraint on g_{uu}

Results: rp-component

The EGB field equations rp-**component** takes the form:

$$
\begin{aligned}
&(D-3)\Theta_{,p} - 2\gamma\kappa\Big[(D-5)(2g^{mn}\Theta_{,m}{}^{s}R_{pn} - {}^{s}R\Theta_{,p}) \\
&\quad - 2(D-3)(D-4)\Theta_{,u}\Theta_{,p} - (D-3)^2(D-4)\Theta^2\Theta_{,p}g_{uu} \\
&\quad - (D-3)(D-4)\Theta\Theta_{,p}g_{uu,r} - (D-4)^2\Theta\Theta_{,p}g^{mn}g_{mn,u} \\
&\quad + (D-4)^2\Theta\Theta_{,m}g^{mn}g_{np,u} - 2(D-3)(D-4)\Theta_{,p}\Theta_{,r}g_{uu} \\
&\quad + (\Theta_{,r} + \Theta^2)(D-4)\big(2(D-3)\Theta_{,p}g_{uu} \\
&\qquad + (D-3)\Theta g_{uu,p} - 2g^{mn}g_{m[p,u||n]}\big)\Big] = 0
\end{aligned}
$$

For *simplicity* and in *analogy with GR* (coordinate freedom) we set:

$$\Theta_{,p} = 0 \qquad \text{that is} \qquad \Theta = \Theta(r,u)$$

Results: rp-component

and then the rp**-component** simplifies to:

$$2\gamma\kappa(D-4)(\Theta_{,r}+\Theta^2)\big((D-3)\Theta g_{uu,p}-2g^{mn}g_{m[p,u||n]}\big)=0$$

- identically satisfied for $\Theta_{,r}+\Theta^2=0$
- combination with the rr-component

$$^{s}\mathcal{R}_{||p}=e^{2\int\Theta dr}\Theta(D-3)h^{kl}\left[(D-3)h_{kl,u||p}-(D-2)h_{kp,u||l}\right]$$

where $^{s}\mathcal{R}$ is the transverse space scalar curvature and $_{||p}$ its compatible covariant derivative

Results: ru-component

Using the above assumptions, ru**-component** takes the form:

$$\frac{1}{2}{}^sR - \Lambda + (D-2)\Big(\Theta_{,u} + \frac{1}{2}g^{kl}g_{kl,u} + \frac{1}{2}\Theta g_{uu,r} + \Theta_r g_{uu} + \frac{1}{2}(D-1)\Theta^2 g_{uu}\Big)$$
$$+ 2\gamma\kappa\Big[(D-4){}^sR\Theta_{,u} + (D-2)(D-3)(D-4)\Theta^2\Theta_{,u}g_{uu}$$
$$+ \frac{1}{2}(D-4)\Theta\big({}^sRg^{kl} - 2{}^sR^{kl}\big)g_{kl,u} + \frac{1}{2}(D-4)\Theta{}^sRg_{uu,r}$$
$$+ (D-3)(D-4)\Theta\Theta_{,u}g^{kl}g_{kl,u} + (D-3)(D-4)\Theta\Theta_{,r}g_{uu}g^{kl}g_{kl,u}$$
$$+ \frac{1}{2}(D-2)(D-3)(D-4)\Theta^3 g_{uu}g_{uur} - \frac{1}{4}(D-3)(D-4)\Theta^2 g^{ij}g^{kl}g_{ik,u}g_{jl,u}$$
$$+ \frac{1}{4}\big({}^sR^2_{ikjl} - 4{}^sR^2_{kl} + {}^sR^2\big) + \frac{1}{2}(D-4){}^sRg_{uu}\big(2\Theta_{,r} + (D-3)\Theta^2\big)$$
$$+ \frac{1}{4}(D-2)(D-3)(D-4)\Theta^2 g^2_{uu}\big((D-1)\Theta^2 + 4\Theta_{,r}\big)\Big] = 0$$

Combination with the previous equations.
Results also for pq, up, and uu components, however, even more messy.

Results: Additional constraints – Weyl type

The **Weyl** tensor frame **irreducible components** are:

$$\Psi_{0^{ij}} = 0$$

$$\Psi_{1T^i} = 0 \qquad \tilde{\Psi}_{1^{ijk}} = 0$$

$$\Psi_{2S} = \frac{D-3}{D-1} P \qquad \tilde{\Psi}_{2T^{(ij)}} = m_i^p m_j^q \frac{1}{D-2} \left({}^S R_{pq} - \frac{g_{pq}}{D-2} {}^S R \right)$$

$$\Psi_{2^{ij}} = 0 \qquad \tilde{\Psi}_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q \, {}^S C_{mpnq}$$

$$\Psi_{3T^i} = m_i^p \frac{D-3}{D-2} V_p \qquad \tilde{\Psi}_{3^{ijk}} = m_i^p m_j^m m_k^q \left(X_{pmq} - \frac{2}{D-3} g_{p[m} X_{q]} \right)$$

$$\Psi_{4^{ij}} = m_i^p m_j^q \left(W_{pq} - \frac{g_{pq}}{D-2} W \right)$$

with

- $X_q \equiv g^{pm} X_{pmq}$ and $W \equiv g^{pq} W_{pq}$
- ${}^S C_{mpnq}$, ${}^S R_{pq}$, and ${}^S R$ encoding the transverse space curvature

Coefficients , P, V_p, X_{pmq} and W_{pq} are ...

Results: Additional constraints – Weyl type

Coefficients , P, V_p, X_{pmq} and W_{pq} are ...

$$P = \left(\tfrac{1}{2} g_{uu,r} - \Theta g_{uu}\right)_{,r} + \frac{{}^S R}{(D-2)(D-3)} - 2\Theta_{,u}$$

$$V_p = -\tfrac{1}{2} g_{uu,rp} - \frac{1}{D-3} g^{mn}\left(g_{m[p,u||n]}\right) + \Theta\left[g_{uu,p} - \tfrac{1}{2} g^{rn} g_{np,u} - \frac{1}{2(D-3)} g^{rn} g_{np,u}\right]$$

$$X_{pmq} = g_{p[m,u||q]}$$

$$W_{pq} = -\tfrac{1}{2} g_{uu||pq} - \tfrac{1}{2} g_{pq,uu} + \tfrac{1}{4} g_{uu,r} g_{pq,u} + \tfrac{1}{4} g^{mn} g_{mp,u} g_{nq,u} - \tfrac{1}{2}\Theta g_{uu} g_{pq,u}$$

In addition to the field equations, we have to employ conditions:

$$\Psi_{2S} = 0 \qquad \tilde{\Psi}_{2T^{(ij)}} = 0 \qquad \tilde{\Psi}_{2^{ijkl}} = 0 \qquad \tilde{\Psi}_{3^{ijk}} = 0$$

We are primarily interested in such RT spacetimes, where the only non-vanishing Weyl component is:

$$\Psi_{4^{ij}} = m_i^p m_j^q \left(W_{pq} - \frac{g_{pq}}{D-2} g^{mn} W_{mn}\right)$$

i.e. **Weyl type N** in $D > 4$

Summary

Robinson–Trautman solutions to **Einstein–Gauss–Bonnet** gravity:

- we have derived the explicit form of the EGB field equations
- we have identified distinct subclasses
- we try to employ additional constraints on the Weyl tensor to find a generic Weyl type N solution in contrast to $D > 4$ GR

  This is almost complete; however, it is still a work in progress.

This talk will be summarised in the upcoming paper:

Robinson–Trautman spacetimes in the Einstein-Gauss-Bonnet theory
N. Astudillo Neira, R. Švarc, hopefully appear soon on arXive

`Thank you for your attention!`