The Common Solution Space of GR

One Equation to Rule Them All

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Based on:

- A.P. The common solution space of General Relativity, J. Geom. Phys (2024)
- A.P. Linearization of Newton's Second Law, Int. J. Theor. Phys. (2025)
- A.P. S. Moyo & P.G.L. Leach, A Geometric Interpretation for the Algebraic Properties of SecondâOrder Ordinary Differential Equations, Math. Meth. Appl. Sci. (2025)
- A.P. Canonical Structure and Hidden Symmetries in Scalar Field Cosmology, (2025) [arXiv:2505.15207]

Arscott emphasized in the preface of his book[1] the significance of pursuing analytic solutions rather than depending solely on numerical methods.

> *"...fall back on numerical techniques savours somewhat of breaking a door with a hammer when one could, with a little trouble, find the key".*

The determination of the integrability properties of dynamical systems is essential in physics and in all areas of applied mathematics. The novelty when a physical system is described by an integrable dynamical system is that we know that an actual solution exists when we apply numerical methods for the study of the system, or there exists a closed-form function which solves the dynamical system.

[1] F.M. Arscott, Periodic Differential Equations, Pergamon Press, (1964)

The precise meaning of the solution for a system of differential equations can be cast in several ways. Three of these are:

- A set of explicit functions describing the variation of the dependent variables with the independent variable(s) (Closed-form solutions)
- The existence of a sufficient number of independent explicit first integrals and invariants.
- The existence of a sufficient number of explicit transformations which permits the reduction of the system of differential equations to a system of algebraic equations.

In gravitational physics exact/analytic solutions[2]:

- Allow for the precise modeling of gravitational fields and spacetime structures (Strong field Black holes, Early Universe Inflation, Dark Energy, Deviation from GR in modified gravity).
- Serve as reference points for evaluating numerical simulations and approximation techniques (Verify the accuracy of the approximation especially near singularities[3] i.e. Kasner approximation in the Mixmaster, give insing on the initial value problem).
- Reduce the necessary computing power. (perturbations, data analysis).

[2]H. Stephani et al, Exact Solutions of Einstein's Field Equations (2003)

[3]M.A.H. MacCallum, Exact solutions in cosmology (1984)

Exact solutions of special interests:

- Isotropic and homogeneous Cosmology: de Sitter Universe; ΛCDM; Milne Universe.
- Anisotropic and homogeneous Cosmology: Kasner; Bianchi III (Kantowski-Sachs); Bianchi VII_h.
- Inhomogeneous Cosmology: LTB; Stephani metric; Barnes metric; Szekeres spacetimes.
- Black holes: Schwarzschild; De Sitter-Schwarzschild; Reissner-Nordström; Kerr.

The Jacobi metric and the Eisenhart lift are distinct methods for geometrically representing dynamical systems. Notably, autonomous dynamical systems can be formulated as a set of geodesic equations. In this investigation, we focus on the Eisenhart lift, a technique that involves augmenting the dimensionality of the dynamical system. Specifically, this geometrization process entails introducing additional dimensions through the inclusion of new dependent variables. A novel kinetic metric is introduced, characterized by at least one isometry associated with a Noetherian conservation. When this isometry is applied, the geodesic equations are reduced back to the original dynamical system.

Consider the Hamiltonian Function $H \equiv \frac{1}{2}p_x^2 + V(x) = h$.with equations of motion $\dot{x} = p_x, \ \dot{p}_x = -V_{,x}$.

Via the Eisenhart lift we can introduce the equivalent Hamiltonian $H_{1+1} \equiv \frac{1}{2}p_x^2 + \frac{\alpha}{2}V(x)\,p_z^2 = h_{1+1}$.which describes the geodesic equations of the 2D space with line element $ds_{(1+1)}^2 = dx^2 + \frac{1}{\alpha V(x)}dz^2$, and equations

$$\begin{aligned} \dot{x} &= p_x \ , \ \dot{p}_x = -\frac{\alpha}{2}V_{,x}p_z^2 \\ \dot{z} &= \alpha V(x)\,p_z, \ \dot{p}_z = 0. \end{aligned}$$

The solution of the latter is a solution of the original system iff $h_{1+1} = h$ and $\alpha\left(p_z^0\right)^2 = 2$.

The lift is not unique. Alternative lifts are

$$H_{1+2} \equiv \frac{1}{2}p_x^2 + V(x)\, p_u^2 + p_u p_v = h_{n+2},\;\; p_u p_v - h_{n+2} = h$$

$$\begin{aligned} H_{1+3} &= \frac{1}{2}p_x^2 + \frac{\alpha}{2}F_1(x)\, p_z^2 + F_2(x)\, p_u^2 + p_u p_v = h_{1+3}, \\ V(x) &= \frac{\alpha}{2}F_1(x)\, p_z^2 + F_2(x)\, p_u^2 + p_u p_v \;,\;\; p_u p_v - h_{1+3} = h \end{aligned}$$

and many other...

We can introduce the equivalent system
$H \equiv \frac{1}{2}p_x^2 + V(x) - h = 0$, such that
$U(x) = V(x) - h$, $U_{,x} = V_{,x}$ Then, the extended Hamiltonian reads

$$\tilde{H}_{1+1} \equiv \frac{1}{2}p_x^2 + \frac{\alpha}{2}U(x)\,p_z^2 = 0,$$

with the same equations of motion, but energy zero (null geodesics). However, null geodesics are invariant under conformal transformations, that is, the solution is the same and for the singular Hamiltonian system

$$\tilde{H}_{1+1} \equiv N^2\left(\frac{1}{2}p_x^2 + \frac{\alpha}{2}U(x)\,p_z^2\right) = 0,$$

For the 1D original system H, the extended Hamiltonian $\tilde{H}_{1+1}$ describes the null geodesics of the 2D space with line element $ds = dx^2 + \frac{1}{\alpha U(x)} dz^2$. All 2D spacetimes are conformally flat. Indeed, under the change of variables $dx = \left(U\left(\int F(x)\,dx\right)\right)^{-\frac{1}{2}} dX$, the line element

$$ds^2_{(1+1)} = \left(U\left(\int F(x)\,dx\right)\right)^{-1}\left(dx^2 + dz^2\right).,\tag{1}$$

Consequently, the Hamiltonian reads $\tilde{H}_{1+1} \equiv U\left(\int F(X)\,dX\right)\left(p_X^2 + \frac{1}{\alpha} p_z^2\right) = 0$. The equations of motion read $\ddot{X} = 0,\ \ddot{z} = 0$.

The minisuperspace description is essential within gravitational theories, as it allows us a better understanding of the dynamics and permits a physical description on geometrodynamical terms.

- The gravitational field equations can be expressed in a more simplified manner so as to give to certain degrees of freedom[4].
- The establishment of a point-like Lagrangian function, facilitates the utilization of mathematical tools and techniques from analytical mechanics.
- The equivalent Hamiltonian formulation, through the use of the Dirac-Bergmann algorithm[5] allows the distinction of the true degrees of freedom for the reduced gravitational system. This is useful, especially in cases where the generic Hamiltonian formalism of GR.

[4]M.P. Ryan and L.C. Shepley, Homogeneous Relativistic Cosmologies, Princeton University Press (1975)

[5]P.A.M. Dirac, Canad. J. Math 2 (1950); J. Anderson and P. Bergmann, PR. 83 (1951)

Consider the line element

$$ds^2 = -a^2(r)\,dt^2 + n^2(r)\,dr^2 + b^2(r)\left(d\theta^2 + \sin^2\theta\ d\phi^2\right).$$

Only two of the three scale factors $a(r)$, $b(r)$ and $n(r)$ are essential, and they are determined by the solution of the field equations.

In the presence of Λ the solution of Einstein's field equations gives the de Sitter-Schwarzschild solution

$$ds^2 = -\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2 d\Omega^2.$$

The point-like Lagrangian which describes the evolution of the scale factors, leading to the analytic solution is

$$L^{\Lambda}\left(n, a, a', b, b'\right) = \frac{1}{2n}\left(8ba'b' + 4ab'^2\right) + 2na\left(1 + \Lambda b^2\right).$$

We introduce the equivalent Hamiltonian function in the Eisenhart-Duval lift formalism

$$\mathcal{H}^{\Lambda} = n\left(\frac{p_a p_b}{4b} - \frac{a}{8b^2}p_a^2 - 2a\left(1+\Lambda b^2\right)p_\psi^2\right),$$

with constraints $\mathcal{H}^{\Lambda}=0$ and $p_\psi = 1$.

The Cotton-York tensor for the extended minisuperspace has zero components, thus the space is conformally flat. We define the new variables $a=\sqrt{\frac{A}{b}}$, $dB = b+\frac{\Lambda}{3}b^3$ and $A=\frac{X+Y}{2\sqrt{2}}$, $B=\frac{X-Y}{2\sqrt{2}}$, $\hat{n}=\left(1+\Lambda b^2\right)n\left(\frac{b}{A}\right)^{\frac{1}{2}}$.

The extended minisuperspace reads
$ds^{\Lambda\ 2} = \frac{1}{2\hat{n}}\left(dX^2 - dY^2 - d\psi^2\right)$. The field equations are written in the equivalent form of the free particle, i.e.
$X'' = 0\ ,\ \ Y'' = 0\ ,\ \ \psi'' = 0$,with $X'^2 - Y'^2 - \psi'^2 = 0\ ,\ \ p_\psi = 1$.

The Schwarzschild black hole, share a common solution space, which is that of the null geodesic equations in a conformally flat extended minisuperspace. It's important to note that this transformation does not relate the physical space but rather the space of solutions for the scale factors of spacetime.

With this approach we found that the solution space for the field equations of the following gravitational models is that of the geodesic equations for the flat space.

- GR Vacuum within LRS Bianchi II, LRS Bianchi V & LRS Bianchi VI background.
- Szekeres solution with or without cosmological constant.
- Spatially flat FLRW $S=\int\sqrt{-g}d^4x\left[R-\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}+V\left(\phi\right)\right]$, for the exponential potential $V\left(\phi\right)=V_0e^{-\lambda\phi}$, and its conformal equivalent theories.
- Spatially flat FLRW quintessence scalar field with Chameleon mechanism and an ideal gas, for exponential potential and exponential coupling.
- In modified theories.

The common feature of these geometric linearizable gravitational models is the presence of the $D \otimes_s T_2$ Lie algebra. The origin of this algebra is on the elements of the conformal algebra for the conformally flat extended minisuperspace. The application of the constraint $p_\psi = 1$, in order to determine the original system, indicates that the elements of $D \otimes_s T_2$ remain local symmetries.

Indeed, when a two-dimensional constraint Hamiltonian system is invariant under point transformations with generators the elements of the $D \otimes_s T_2$ Lie algebra, the dynamical system can be linearized, and the closed-form solution of the field equations can be written in analytic form given by the solution of the free particle.

Consider the minisuperspace Lagrangian for the ΛCDM model

$$L(N, a, \dot{a}) = \frac{3}{N} a \dot{a}^2 + N\left(2\Lambda a^3 + \rho_{m0}\right).$$

We observe that when $a \simeq r^{\frac{2}{3}}$, the gravitational field equations are equivalent with the "oscillator". When $\Lambda = 0$, we end with the equation for the free particle.

Is there a transformation which relates the two solutions? Lie's transformation fails because does not preserve the constraint equation.

We introduce the extended Hamiltonian function

$$\mathcal{H}_{lift}^{\Lambda} \equiv N\left(\frac{p_a^2}{12a} + p_u p_v - \left(\omega a^3 + \mu\right) p_v^2\right) = 0.$$

The equations of motion are

$$\frac{1}{N}\dot{a} = \frac{p_a}{6a}\ ,\ \ \frac{1}{N}\dot{u} = p_v - 2p_u\ ,\ \ \frac{1}{N}\dot{v} = p_u,$$

$$\frac{1}{N}\dot{p}_a = 3\omega a^2 p_u\ ,\ \ \frac{1}{N}\dot{p}_u = 0\ ,\ \ \frac{1}{N}\dot{p}_v = 0,$$

in which $p_u = p_u^0$ and $p_v = p_v^0$ are conservation laws. The original ΛCDM model is recovered when

$$\rho_{m0} = \mu p_v^0 - p_u^0 p_v^0 \text{ and } \Lambda = \frac{\omega}{2}\left(p_u^0\right)^2.$$

Consider now the extended Hamiltonian for the *CDM* model,

$$\mathcal{H}_{lift}^{CDM} \equiv N\left(\frac{p_a^2}{12a} + p_u p_v - \mu p_v^2\right) = 0,$$

with constraint $\rho_{m0} = \mu p_v^0 - p_u^0 p_v^0$.

The WdW equation is calculated

$$\frac{1}{6a}\left(\frac{\partial^2}{\partial a^2} - \frac{1}{2a}\frac{\partial}{\partial a}\right)\Psi + \frac{\partial}{\partial v}\left(\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v}\right)\Psi = 0$$

When we apply the operators
$\left(i\frac{\partial}{\partial u}+p_u^0\right)\Psi=0$, $\left(i\frac{\partial}{\partial v}+p_v^0\right)\Psi=0$, we calculate the wavefunction$\Psi=\bar{\Psi}\left(a\right)e^{-i\left(p_u^0u+p_v^0v\right)}$ with

$$\bar{\Psi}\left(a\right)=\bar{\Psi}_1\exp\left(4\sqrt{\frac{\rho_{m0}}{3}}a^{\frac{3}{2}}\right)+\bar{\Psi}_2\exp\left(-4\sqrt{\frac{\rho_{m0}}{3}}a^{\frac{3}{2}}\right)$$

$\rho_{m0}=\mu\left(p_v^0\right)^2-p_u^0p_v^0$. The wavefunction is written in the form $\Psi\simeq e^{-iS_{CDM}(a,u,v)}$, where function $S_{CDM}\left(a,u,v\right)$

$$S_{CDM}\left(a,u,v\right)=\pm4\sqrt{\frac{\rho_{m0}}{3}}a^{\frac{3}{2}}+\left(p_u^0u+p_v^0v\right), \qquad (2)$$

which is the action for the CDM model.

The WdW equation is invariant under the transformation with generator [6]

$$X = 2ua\frac{\partial}{\partial a} + 3\left(1+u^2\right)\frac{\partial}{\partial u} + \left(2\frac{\mu}{\beta} + 3 + 6\mu - \left(3\mu u^2 + 2a^3\right)\right)\frac{\partial}{\partial v}.$$

In the canonical variables

$$a = \frac{\alpha}{\cos\left(\sqrt{\frac{3}{2}\beta}U\right)^{\frac{2}{3}}}\ ,\ u = \frac{1}{\sqrt{\mu}}\tan\left(\sqrt{\frac{3}{2}\beta}U\right)\ , \tag{3}$$

$$v = \sqrt{\frac{2\mu}{3\beta}}\left(\mu U + V\right) - \frac{\sqrt{\mu}}{3}\left(3+4\alpha^3\right)\tan\left(\sqrt{\frac{3}{2}\beta}U\right), \tag{4}$$

the vector field reads $X = \partial_V$ and the WDW equation

$$\frac{1}{6\alpha}\left(\frac{\partial^2}{\partial\alpha^2} - \frac{1}{2\alpha}\frac{\partial}{\partial\alpha}\right)\Psi + \frac{\partial}{\partial V}\left(\frac{\partial}{\partial U} + \left(\mu - 2\beta\alpha^3\right)\frac{\partial}{\partial V}\right)\Psi = 0,$$

[6] A.P. arXiv:2405.20683

We define the quantum operators $\left(i\frac{\partial}{\partial U}+p_U^0\right)\Psi=0$, $\left(i\frac{\partial}{\partial V}+p_V^0\right)\Psi=0$, and the wavefunction is derived $\Psi\left(\alpha,U,V\right)=\hat{\Psi}\left(\alpha\right)e^{-i\left(p_U^0U+p_V^0V\right)}$.

In the WKB approximation, $\hat{\Psi}\left(\alpha\right)\simeq e^{iS_\Lambda(\alpha)}$, we end with the Hamilton-Jacobi equation for the classical limit

$$\frac{1}{12\alpha}\left(\frac{\partial S_\Lambda}{\partial\alpha}\right)^2-\left(2\Lambda\alpha^3+\rho_{m0}\right)=0,$$

with constraints $2\Lambda=-\beta p_V^0$ and $\rho_{m0}=\mu\left(p_v^0\right)^2-p_u^0p_v^0$. This is the Hamilton-Jacobi equation for the ΛCDM model with scale factor $\alpha\left(t\right)$. In a similar way we can apply similar transformations and construct new solutions for the WDW equation which describe universes with different value of Λ.

The above gravitational models have common solution space, that is, the equations of motions for the free particle. Consequently they share the same extended WdW equation expressed in different coordinate system.

The relaxation of the first-class constraints of quantum cosmology can be used to construct different solutions via equivalence transformation and the different gravitational models.

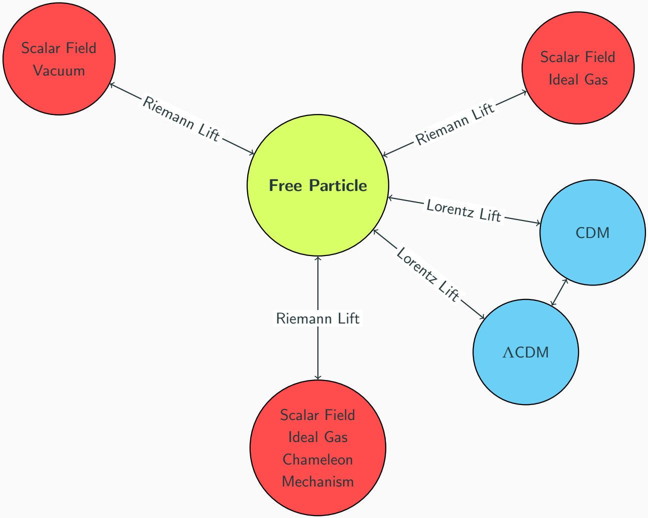

Figure 1: Canonical structure for the solution space between the different cosmological models