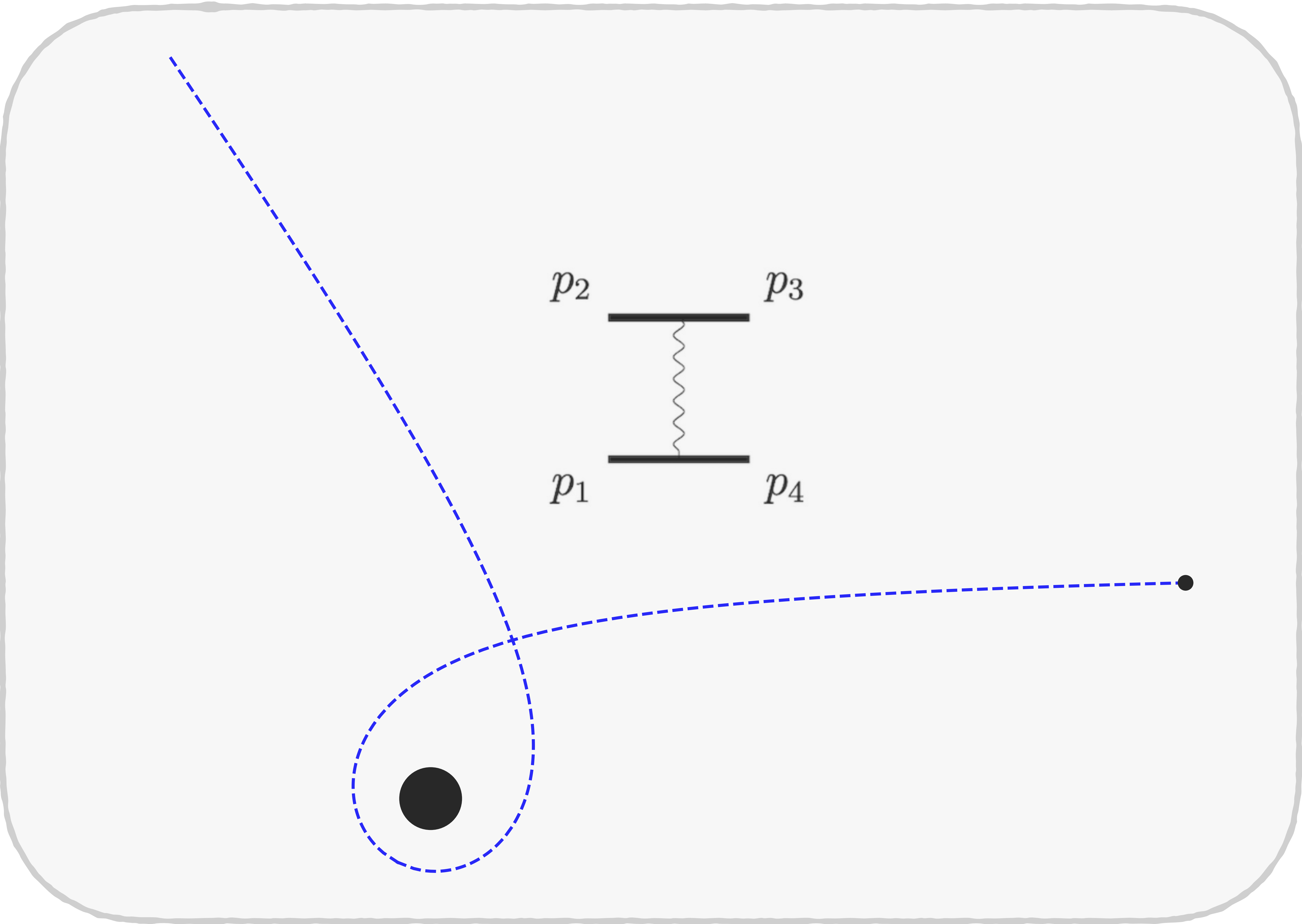
Self-force in hyperbolic scattering



Leor Barack

University of Southampton

3 September 2025 - NEB 21 - Corfu

Plan

- Self-force basics
- Why study black-hole scattering?
- Scattering calculations in a scalar-charge toy model
- Scattering calculations for black holes (prelim)
- Fresh ideas & prospects

credit: **Oliver Long**, visualisation of SXS:BBH:4292

The self-force method

$$G_{\mu\nu}(g_{\alpha\beta}) = 0 \quad / \quad g_{\alpha\beta} = g^{\text{KERR}}_{\alpha\beta} + \varepsilon h^{(1)}_{\alpha\beta} + \varepsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

SELF-FORCE THEORY

from PDEs to point-particle orbits

$$\ddot{x}^\alpha = \varepsilon F^\alpha_{(1)} + \varepsilon^2 F^\alpha_{(2)} + \ldots$$

TWO-TIMESCALE/ADIABATIC EXPANSION

$$\varphi = \varepsilon^{-1} \varphi_{0PA}(\varepsilon t) + \varepsilon^0 \varphi_{1PA}(\varepsilon t) + O(\varepsilon)$$

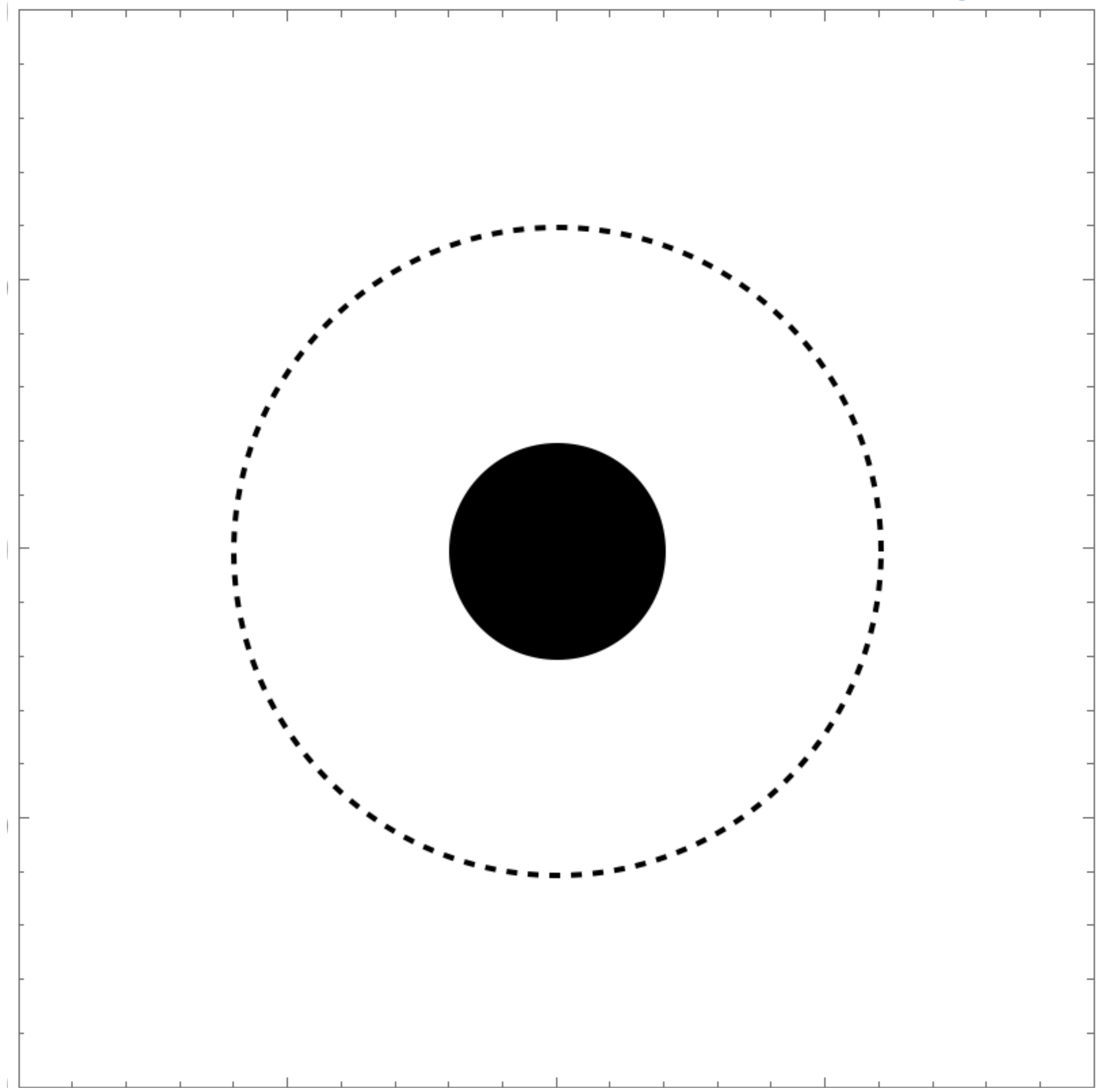
Should suffice for parameter extraction if ε sufficiently small

$$\varepsilon = \frac{m}{M} \ll 1$$

Why scattering?

Main idea: scattering as an efficient probe of strong interaction

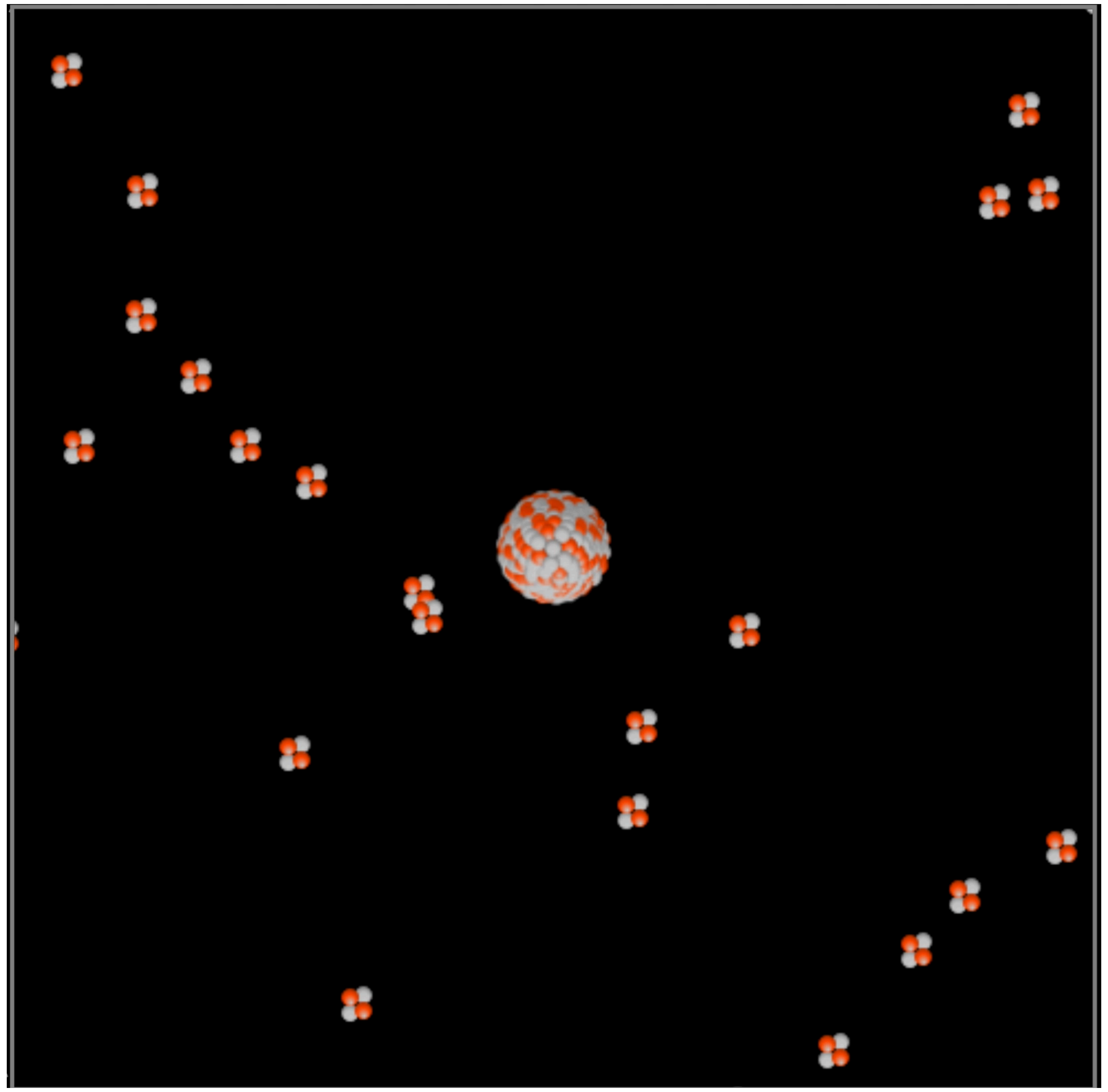
EMRI "zoom-whirl" scattering



credit: Oliver Long

10^{+11} m

Rutherford scattering

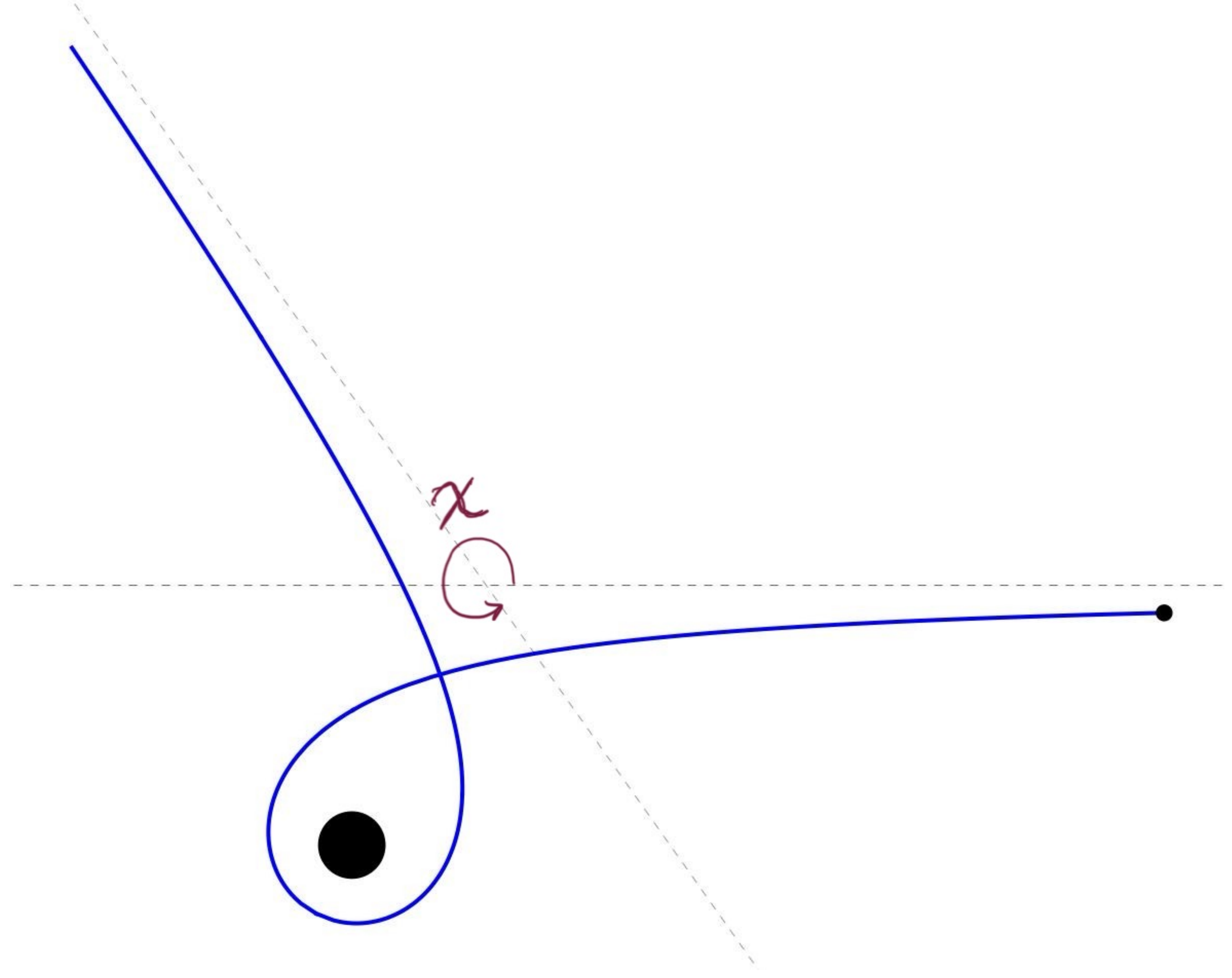


credit: PhET, university of Colorado Boulder

10^{-13} m

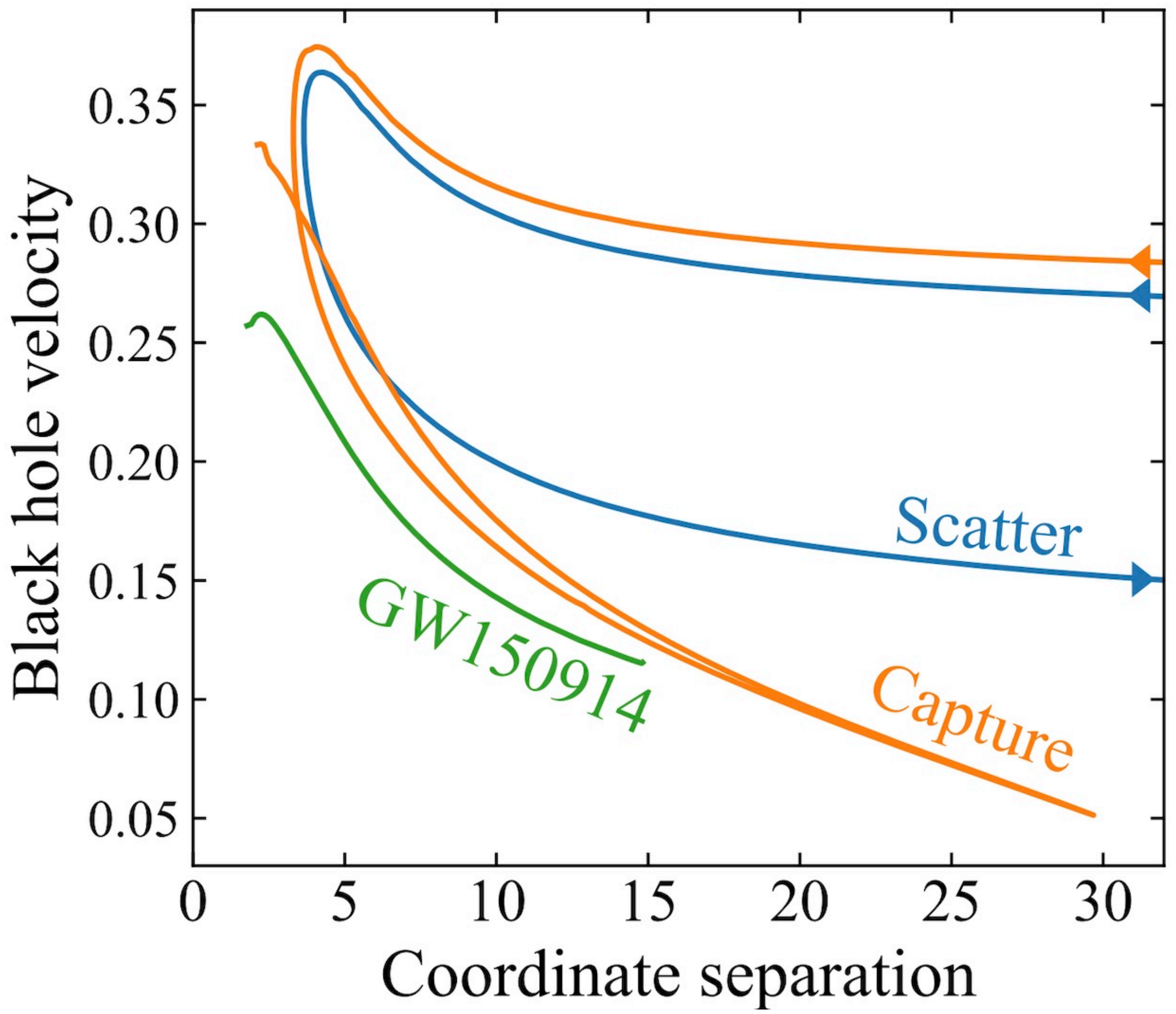
Why scattering?

- Diagnostic "observables" (e.g. scattering angle χ) defined unambiguously from $r \to \infty$ asymptotics.
- Handle on fuller binary parameter space
- $\chi(E, b) \Rightarrow$ full Hamiltonian dynamics
- New way of calibrating EOB theory using post-Minkowskian χ information (Damour 2016)
- "Boundary to bound" maps (Goldberger & Rothstein 2006; Kalin & Porto 2019+)
- Intense cross-disciplinary interest, new participants: EFT, QCD Amplitudes (Bern et al 2019+).



Self-force calculations for scattering:

- ``Easier'' than bound inspiral: no two timescales!
- ``Easy'' access to full high-order PM theory (next slide)
- Access to strong-field dynamics: no weak-field approximation



credit: Oliver Long

Self-force and post-Minkowskian theory

[Damour 2019:] From mass-exchange symmetry and polynomial structure $\Rightarrow$ nSF determines the **full** conservative dynamics to $(2n+2)$PM order (arbitrary ϵ).

$$\chi_{\rm cm} = \frac{E^*_{\rm cm}}{b}\Bigg[\ a_0(v) \qquad \textbf{1PM}$$

$$+\frac{a_1(v)(M+m)}{b} \qquad \textbf{2PM}$$

$$+\frac{a_2(v)(M^2+m^2)+a_{11}(v)Mm}{b^2} \qquad \textbf{3PM}$$

$$+\frac{a_3(v)(M^3+m^3)+a_{21}(v)(M^2m+Mm^2)}{b^3} \qquad \textbf{4PM}$$

$$+\frac{a_4(v)(M^4+m^4)+a_{31}(v)(M^3m+Mm^3)+a_{22}(v)M^2m^2}{b^4}+\cdots\Bigg] \qquad \textbf{5PM}$$

↑ from 0SF! ↑ needs 1SF ↑ needs 2SF

* $E_{\rm cm} = \sqrt{M^2+m^2+2Mm\gamma}$ is initial total energy in initial CoM frame.

Practice problem: (conservative) Self-force effects on the zero-energy zoom-whirl orbit (LB, Colleoni, Damour, Isoyama & Sago 2019)

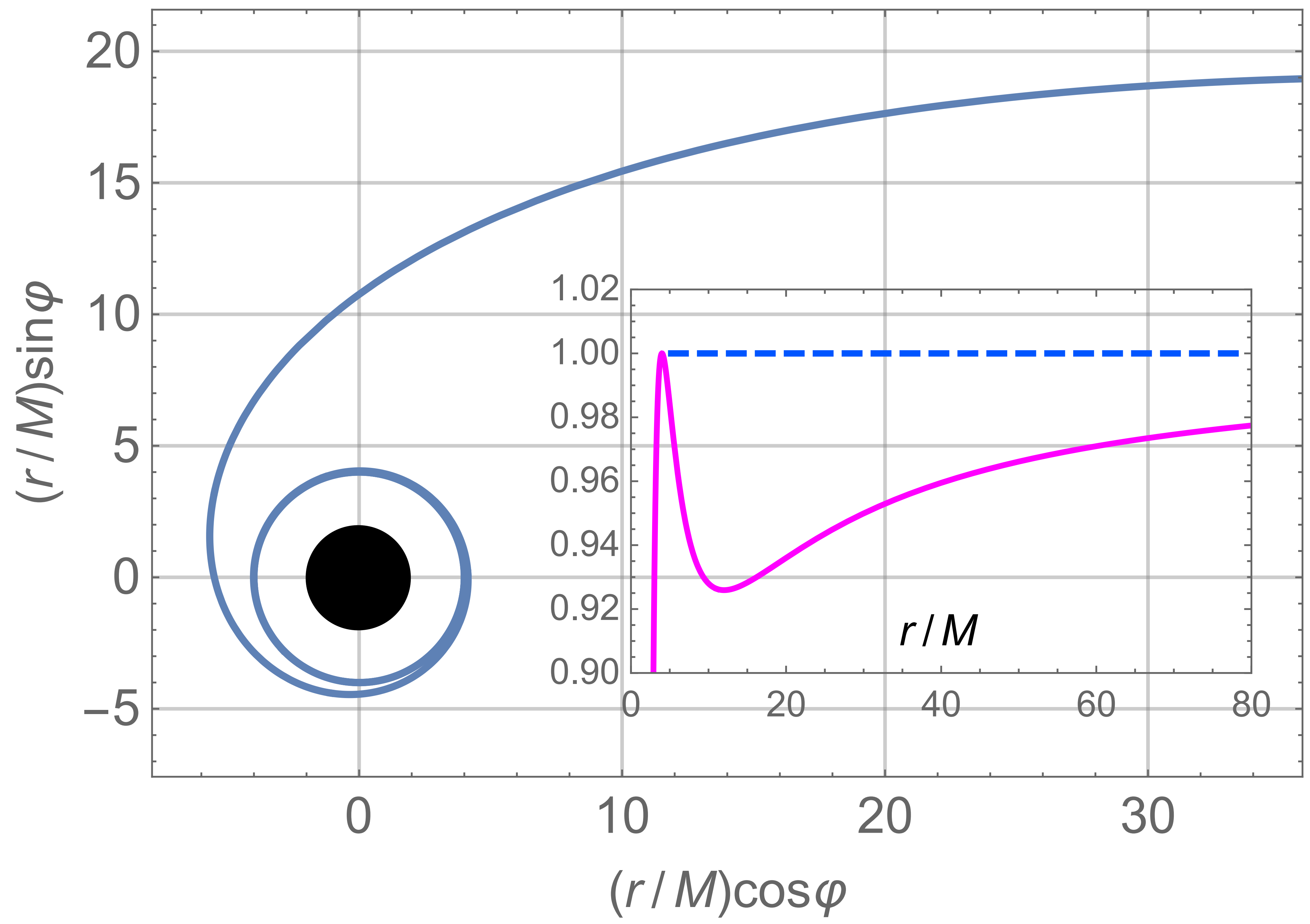
- **Heteroclinic orbit** connects circular orbit to infinity, allowing identification of circular orbit's "binding energy" as a Bondi-type quantity through $O(m^2)$.
- 1SF corrections to $\Omega_{\rm circ}$ and to $L_{\rm crit}$ obtained by integrating self-force along the geodesic orbit:

$$\Omega = (8M)^{-1}\left(1 + 8\epsilon F^r(4M) - 3\epsilon\int_{-\infty}^{\infty} F_t\, d\tau\right)$$

$$L = Mm\left(4 + 4\epsilon - 2\epsilon + \epsilon\int_{-\infty}^{\infty}(F_\varphi - 8F_t)d\tau\right)$$

0SF — recoil — gauge

$E = m, \quad L = 4mM, \quad R = 4M$

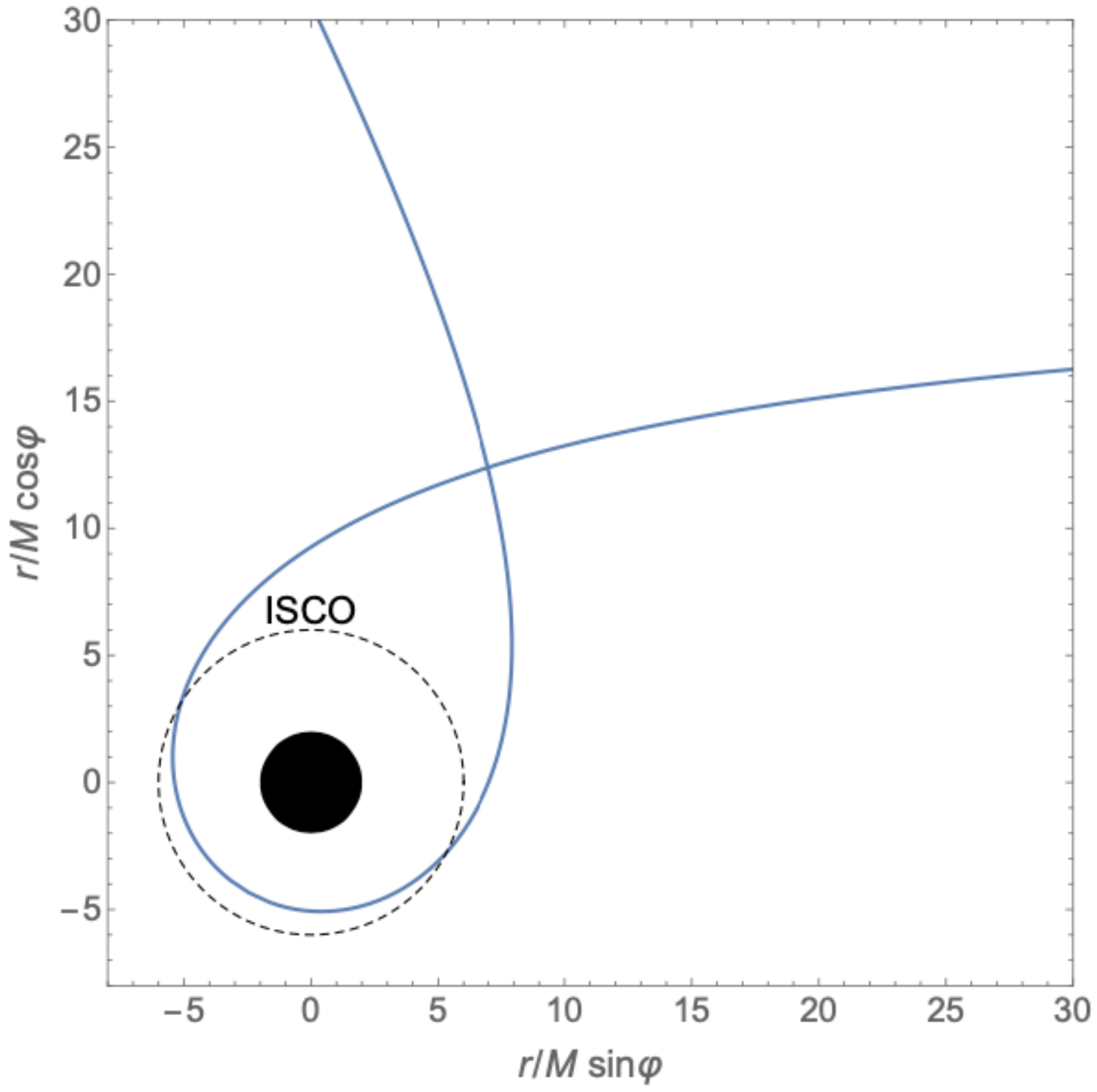


$$\Omega = (8M)^{-1}[1 + 0.5536(2)\epsilon]$$
$$L = 4Mm[1 - 0.304(2)\epsilon]$$

Scattering geodesics preliminaries

parametrisation (Schwarzschild)

- $E := g_{\alpha\beta} t^\alpha u^\beta \Big|_{t\to-\infty} = (1 - v_\infty^2)^{-1/2} = \gamma_\infty$ (specific) Energy
- $L := g_{\alpha\beta} \varphi^\alpha u^\beta \Big|_{t\to-\infty} > L_{\text{crit}}(E)$ (specific) Angular Momentum
- $b := r \sin \Big| \varphi(t) - \varphi(-\infty) \Big|_{t\to-\infty} = \dfrac{L}{\sqrt{E^2 - 1}}$ impact parameter
- $e > 1$ eccentricity
- $p > 6 + e$ semilatus rectum
- $r_{\min} = \dfrac{pM}{1+e} > 3M$ periastron distance



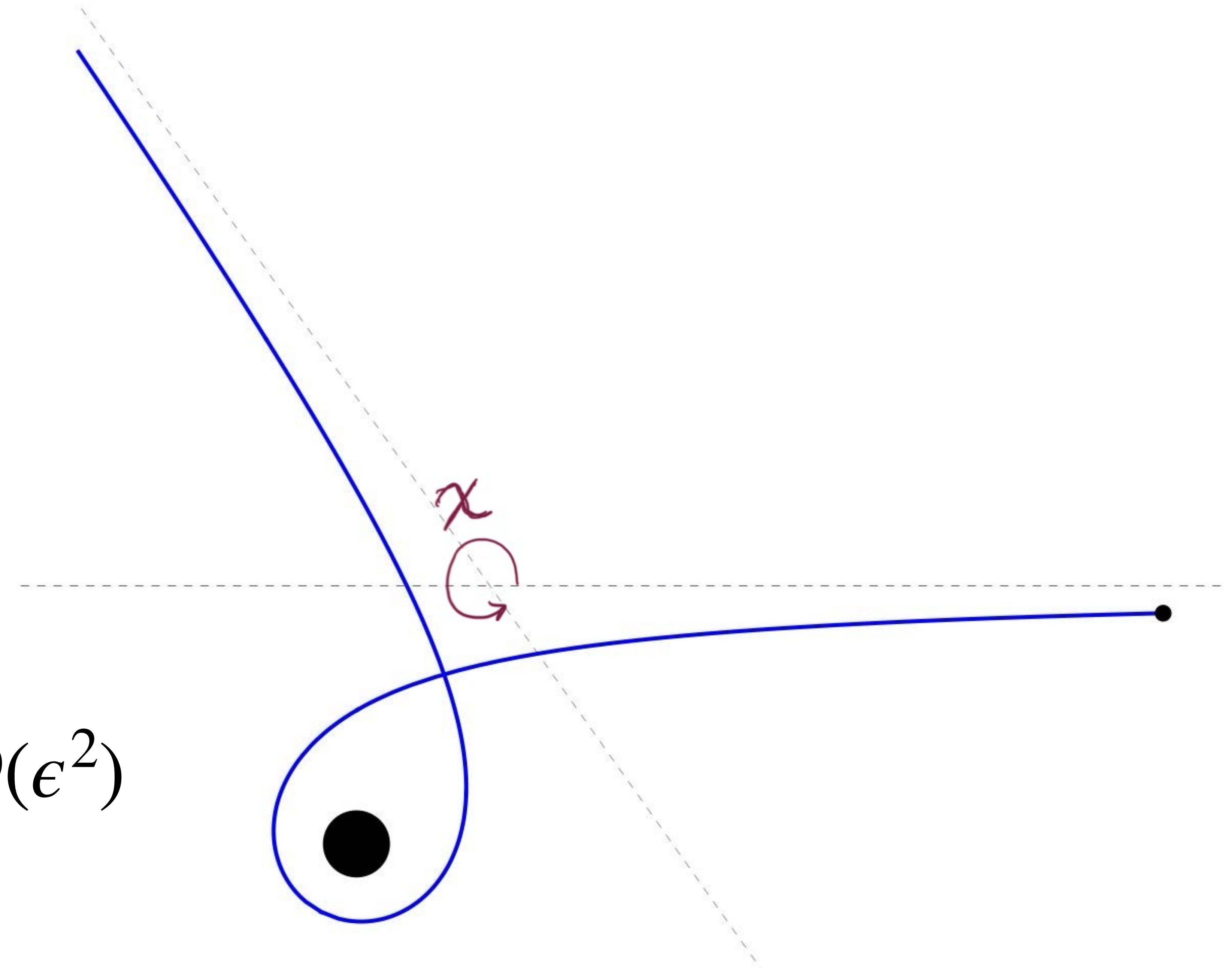
… any two of these!

Scattering orbits preliminaries

"observables"

- Scattering angle:

$$\chi = \int_{-\infty}^{\infty} \frac{d\varphi}{d\tau}(\tau; e, p, F^{\alpha}_{\text{self}})\, d\tau - \pi + (\delta\chi)_{\text{frame}} = \chi_{0\text{SF}} + \chi_{1\text{SF}} + O(\epsilon^2)$$



- Time delay:

$$\Delta t = \int_{-\infty}^{\infty} \left(\frac{dt}{d\tau}(\tau; e, p, F^{\alpha}_{\text{self}}) - \frac{dt}{d\tau}(\tau; e, p, 0) \right) d\tau + (\delta\Delta t)_{\text{frame}} = \Delta t_{1\text{SF}} + O(\epsilon^2)$$

- Radiated energy and angular momentum:

$$E_{\text{rad}} = -\int_{-\infty}^{\infty} F_t^{\text{self}}(\tau; e, p)\, d\tau + (\delta E)_{\text{frame}} \qquad J_{\text{rad}} = \int_{-\infty}^{\infty} F_{\varphi}^{\text{self}}(\tau; e, p)\, d\tau + (\delta J)_{\text{frame}}$$

$$\chi_{1SF}^{\text{diss}} = \alpha(p, e) E_{\text{rad}} + \beta(p, e) J_{\text{rad}}$$ (LB & Long 2022)

Scalar-charge toy model

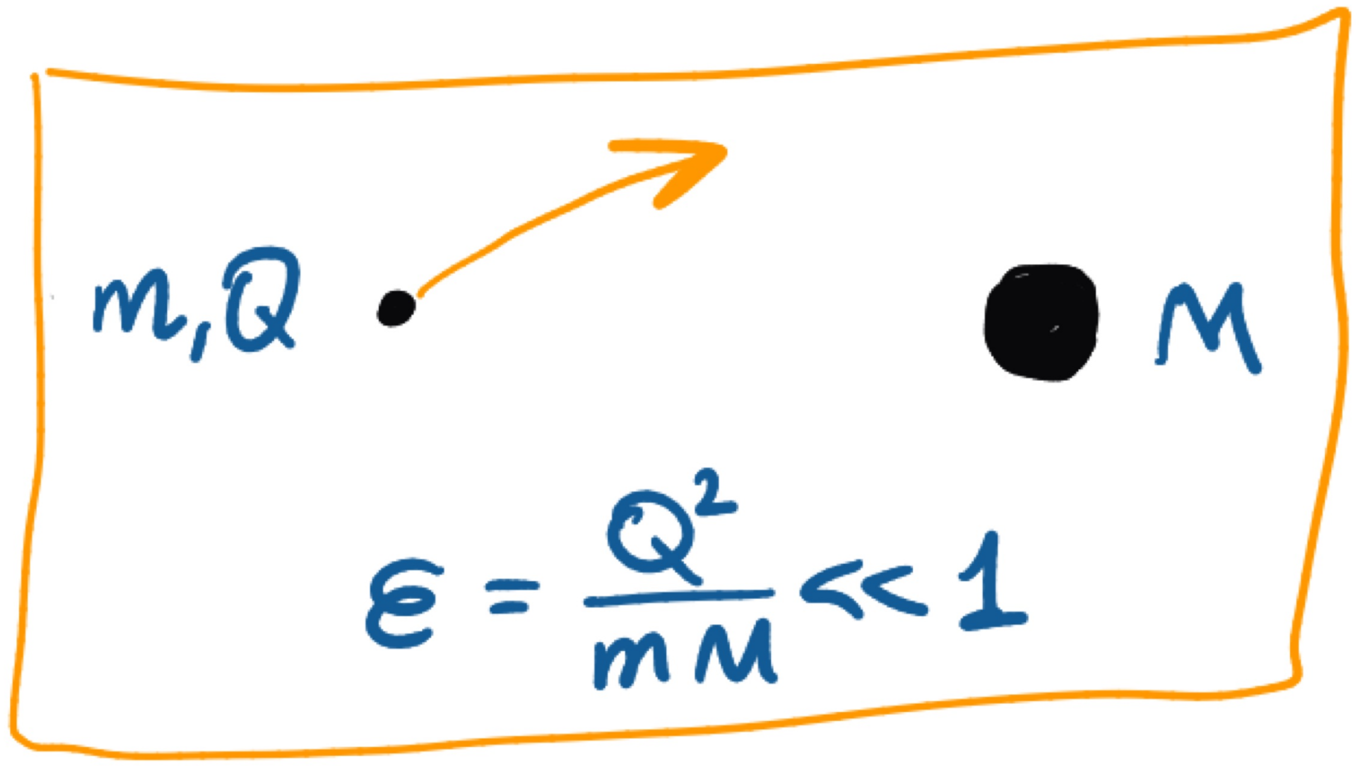
- Charge Sources Klein-Gordon field Φ:

$$\nabla^\alpha \nabla_\alpha \Phi = -4\pi Q \int_{-\infty}^{\infty} (-g)^{-1/2} \delta^4(x - z(\tau)) d\tau$$

- Treat Φ as linear perturbation on Kerr, ignore gravitational self-force, consider only back-reaction from Φ:

$$F^\alpha_{\text{self}} = Q \nabla^\alpha \tilde{\Phi} \propto Q^2$$

- Deviation from geodesic remains small $[O(\epsilon)]$ during scattering, so at leading order can evaluate scattering observables by integrating F_{self} along limiting **geodesic** trajectory.

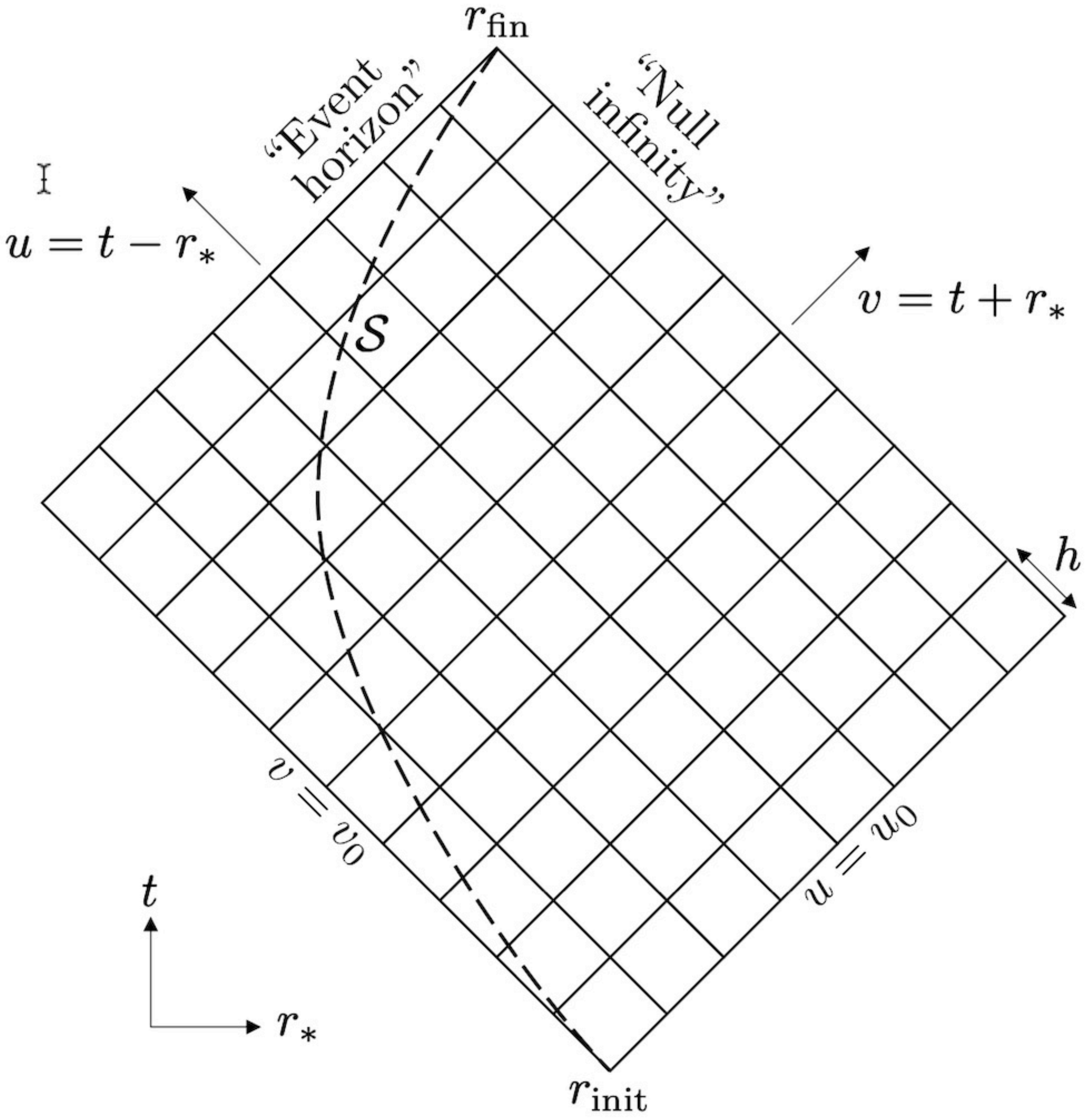


Advantage: Similar mathematical structure, simpler field equation, no frame ambiguities

t-domain numerical method

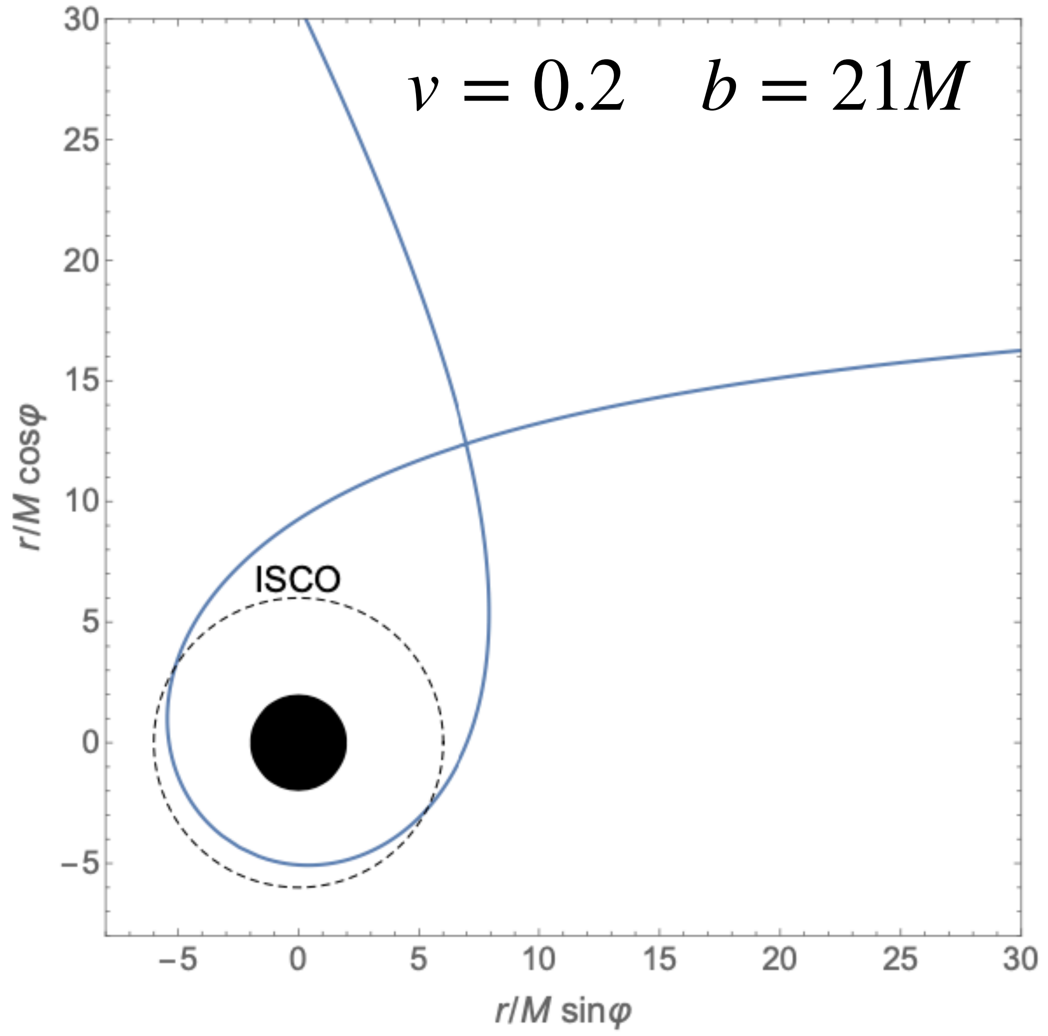
(LB & Long 2022)

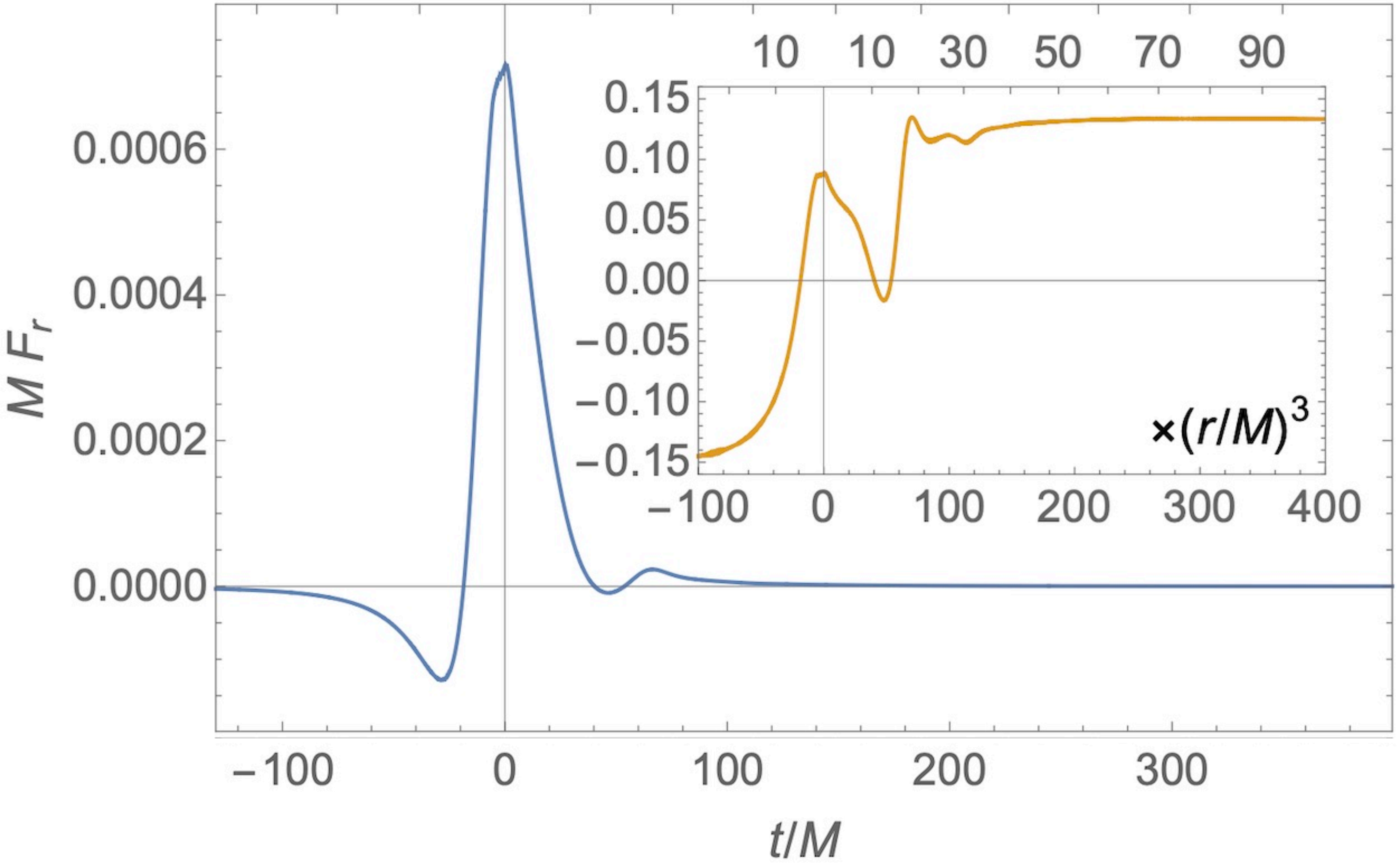
- $\Phi = \frac{Q}{r}\sum_{\ell,m}\phi_{lm}(r,t)Y_{lm}(\theta,\varphi)$
- 1+1D field equation for $\phi_{lm}(r,t)$ discretised and solved in double-null coords.
- zero initial conditions, transient junk discarded
- $F^{\alpha}_{\text{self}}(\tau)$ constructed from ϕ_{lm} using mode-sum regularisation.

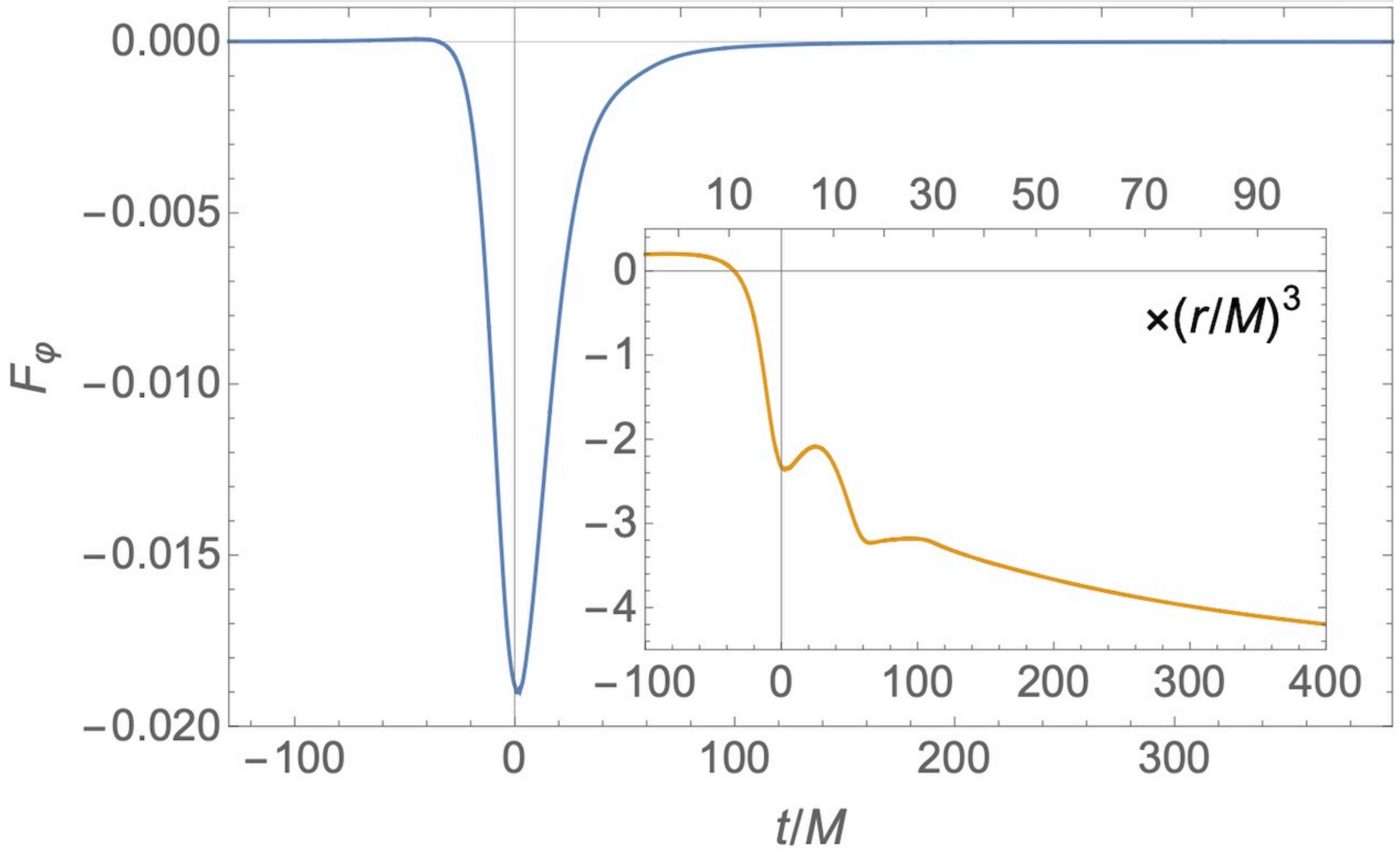


t-domain calculation sample results

(LB & Long 2022)

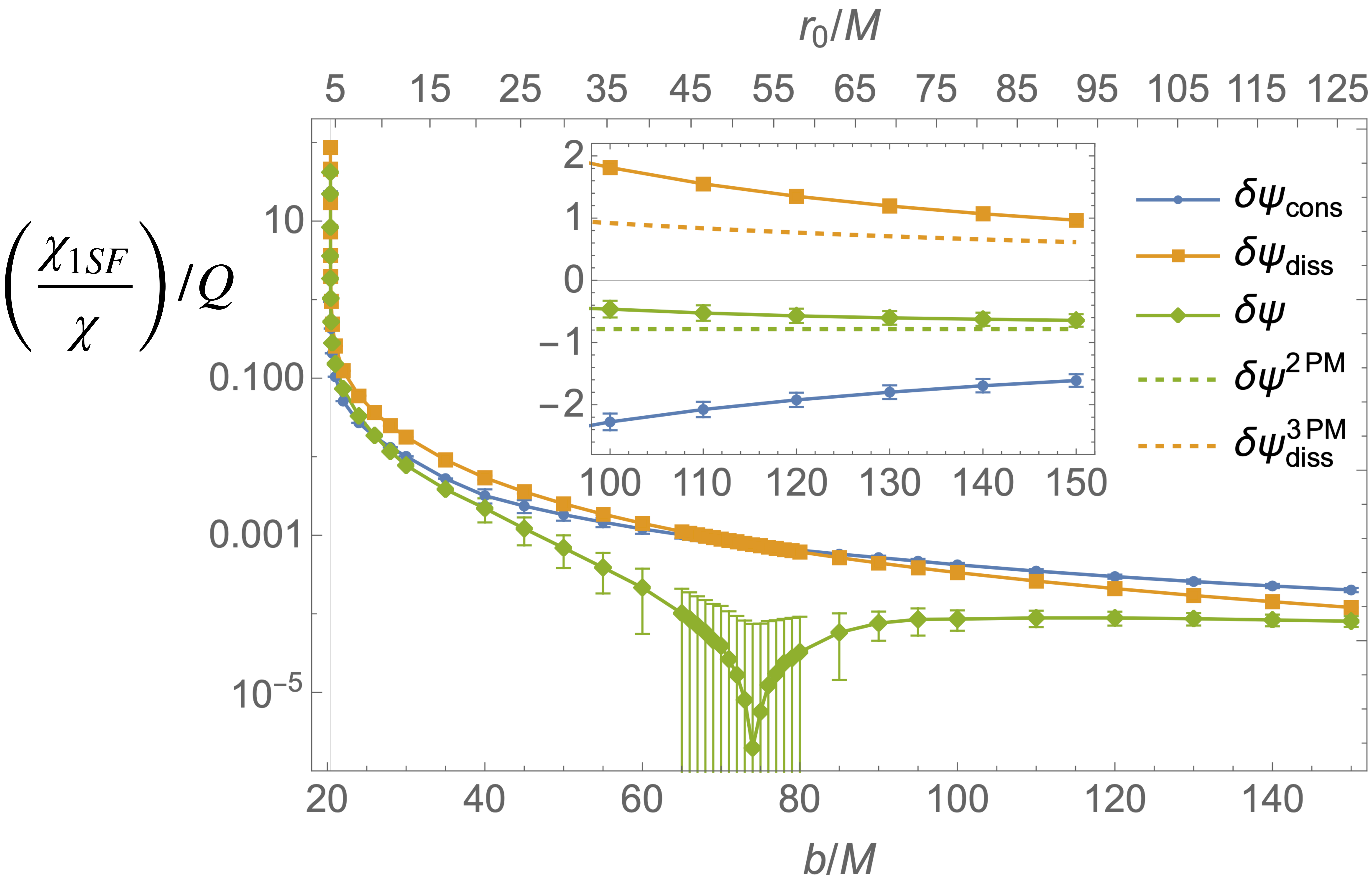






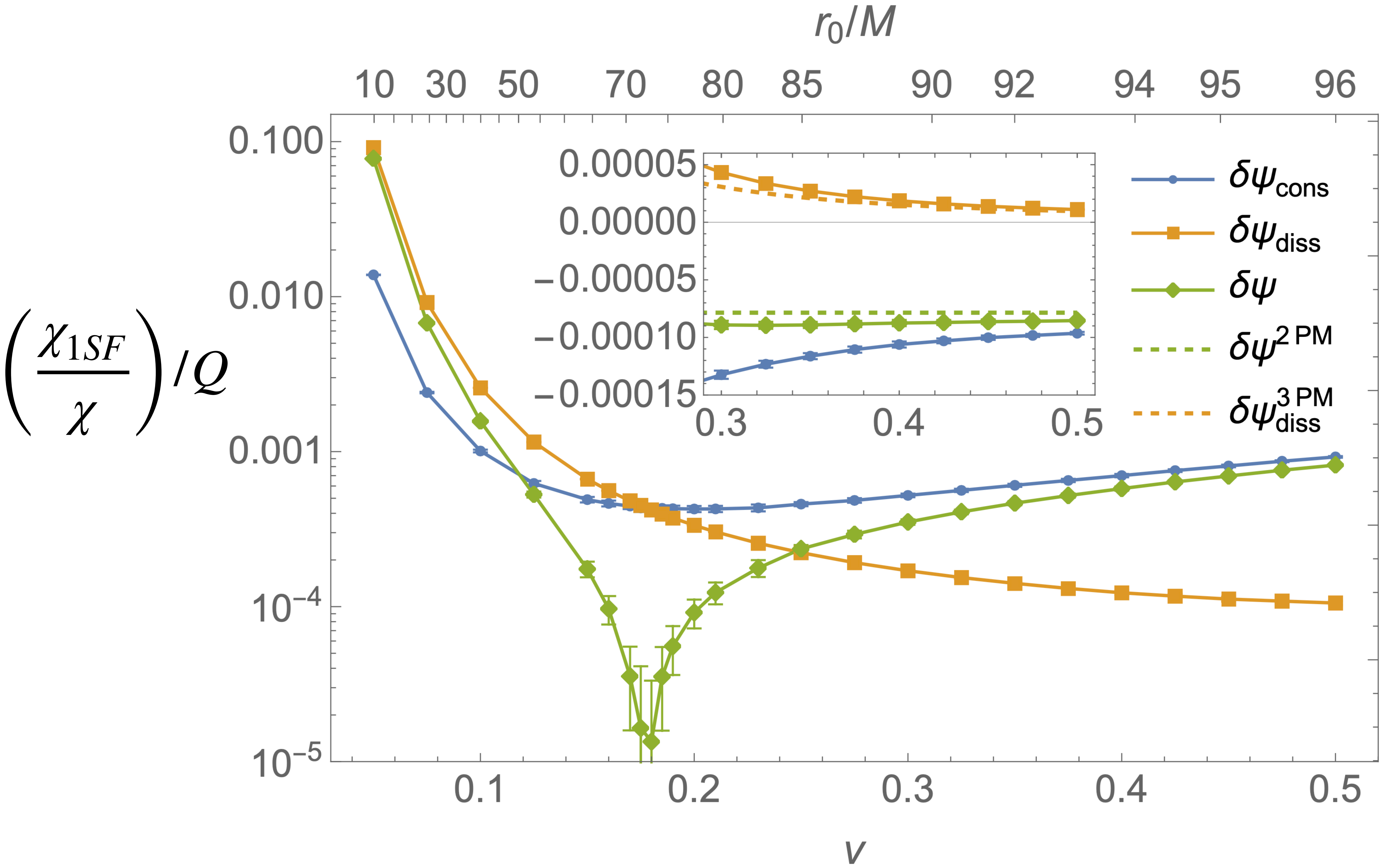
t-domain calculation sample results ($\nu = 0.2$)

(LB & Long 2022)



t-domain calculation sample results ($b = 100M$)

(LB & Long 2022)



Comparison with PM results from Amplitudes

(LB, Bern et al 2023)

Expansion around flat space:

$$\delta\chi^{\text{PM}} = \sum_{i=0}^{\infty} \delta\chi_i \left(\frac{M}{b}\right)^i$$

2PM [Gralla & Lobo '22]:

$$\delta\chi_2^{\text{cons}} = -\frac{\pi}{4}\left(\frac{M}{b}\right)^2 \qquad \delta\chi_2^{\text{diss}} = 0$$

v : Velocity at infinity

b : Impact parameter

3PM:

$$\delta\chi_3^{\text{cons}} = -\frac{4\left(3-v^2\right)}{3v^2\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3 \qquad \delta\chi_3^{\text{diss}} = \frac{2\left(v^2+1\right)^2}{3v^3\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3$$

LO

NLO

4PM dissipative:

$$\delta\chi_4^{\text{diss}} = \left(r_1 + r_2 \,\text{arcsech}\left(\sqrt{1-v^2}\right) + r_3 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{M}{b}\right)^4$$

$r_i =$ rational coefficients

Comparison with PM results from Amplitudes

(LB, Bern et al 2023)

$$
\begin{aligned}
\delta\chi_4^{\text{cons}} =& \Bigg(r_1 + r_2 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) + r_3 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)^2 + r_4 \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2 \\
&+ r_5 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)\mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) + r_6 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2 \\
&+ r_7 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) + r_8 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) \\
&+ r_9 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{10} \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) \\
&+ r_{11} \log^2\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{12}\,\boxed{\alpha} + r_{13}\frac{\boxed{\beta}}{v^2} + r_4 \log(b)\Bigg)\left(\frac{M}{b}\right)^4
\end{aligned}
$$

Elliptic integrals

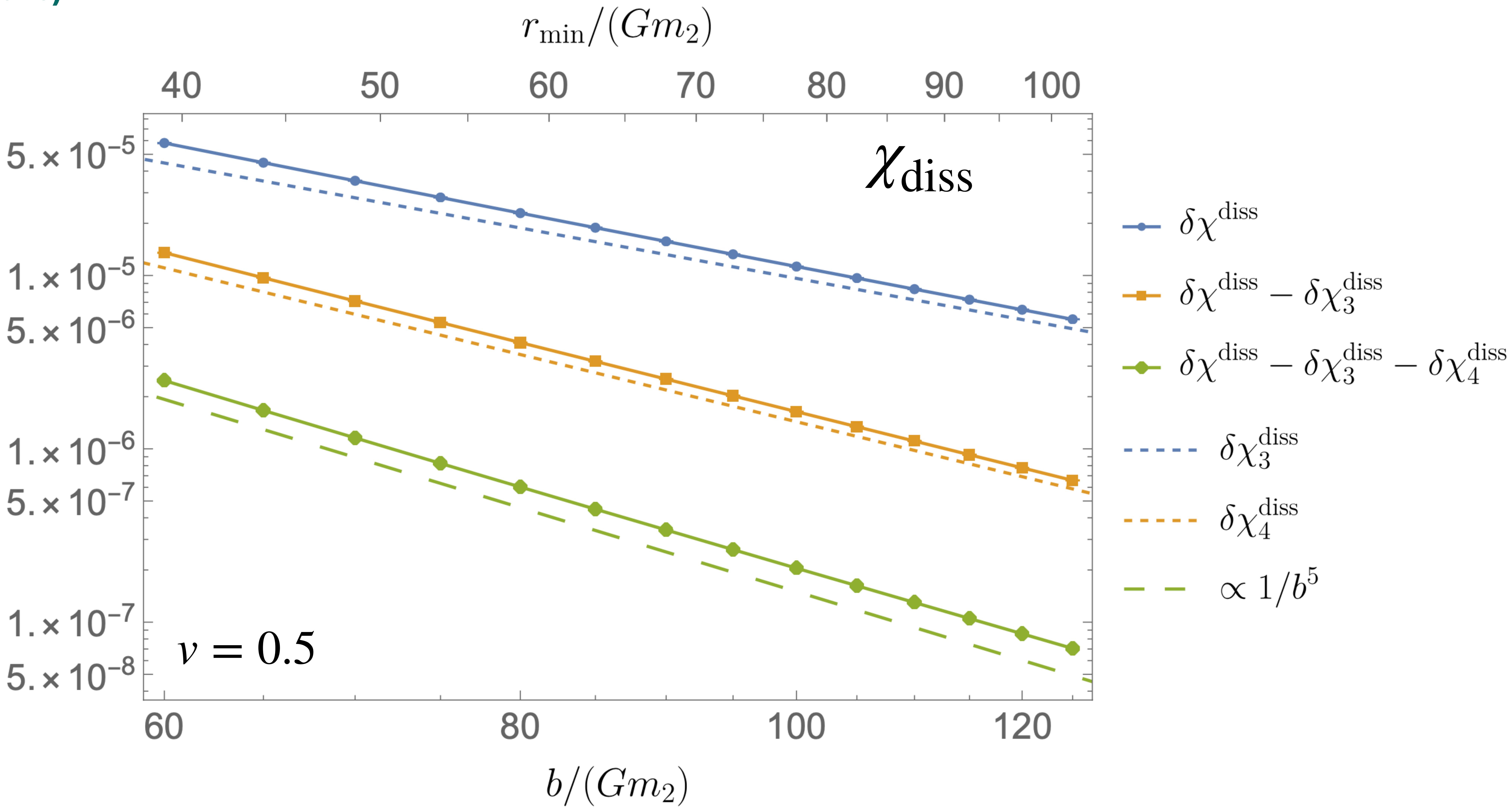
Free coefficients

Log term

$r_i =$ rational coefficients

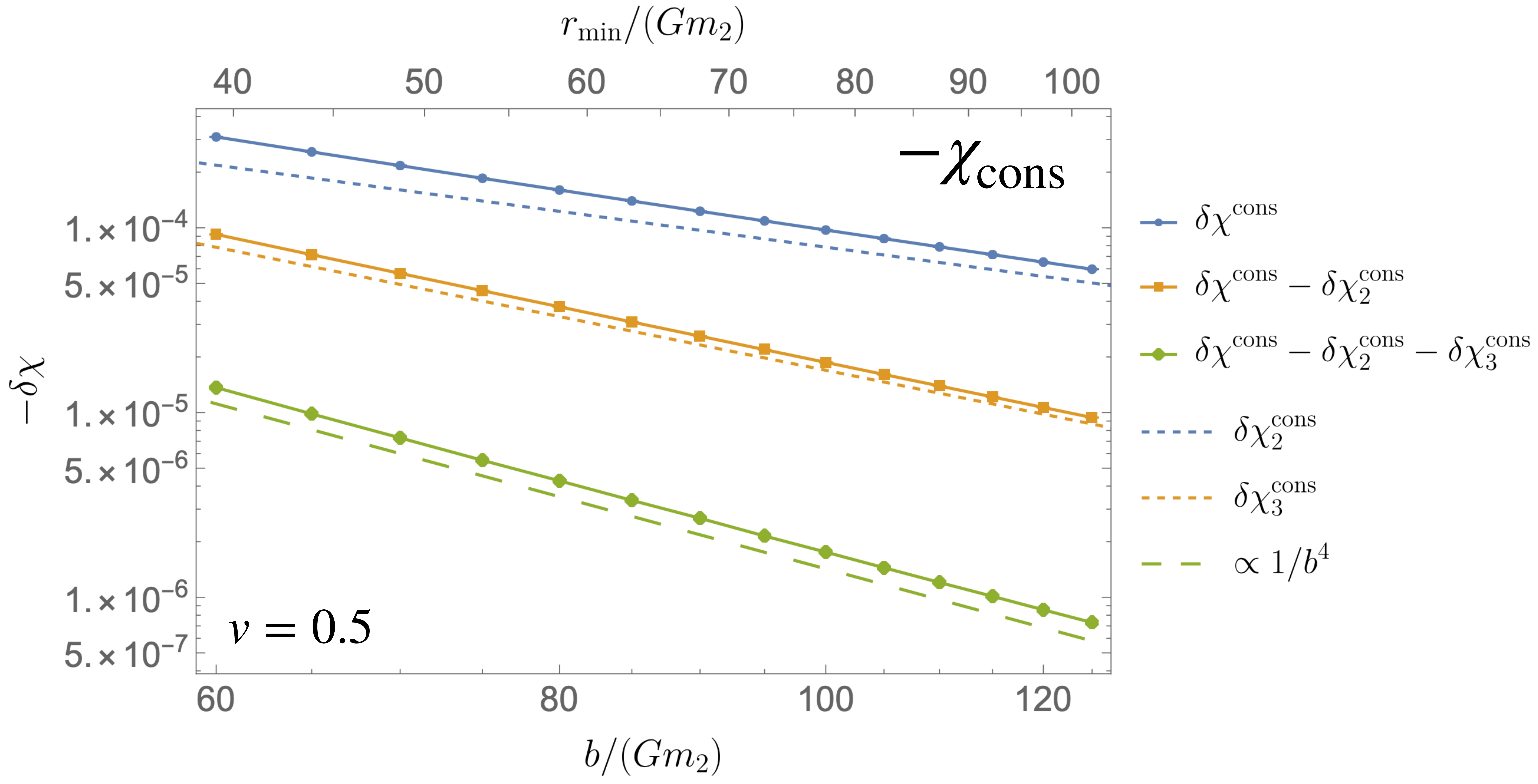
Comparison with PM results from Amplitudes

(LB, Bern et al 2023)



Comparison with PM results from Amplitudes

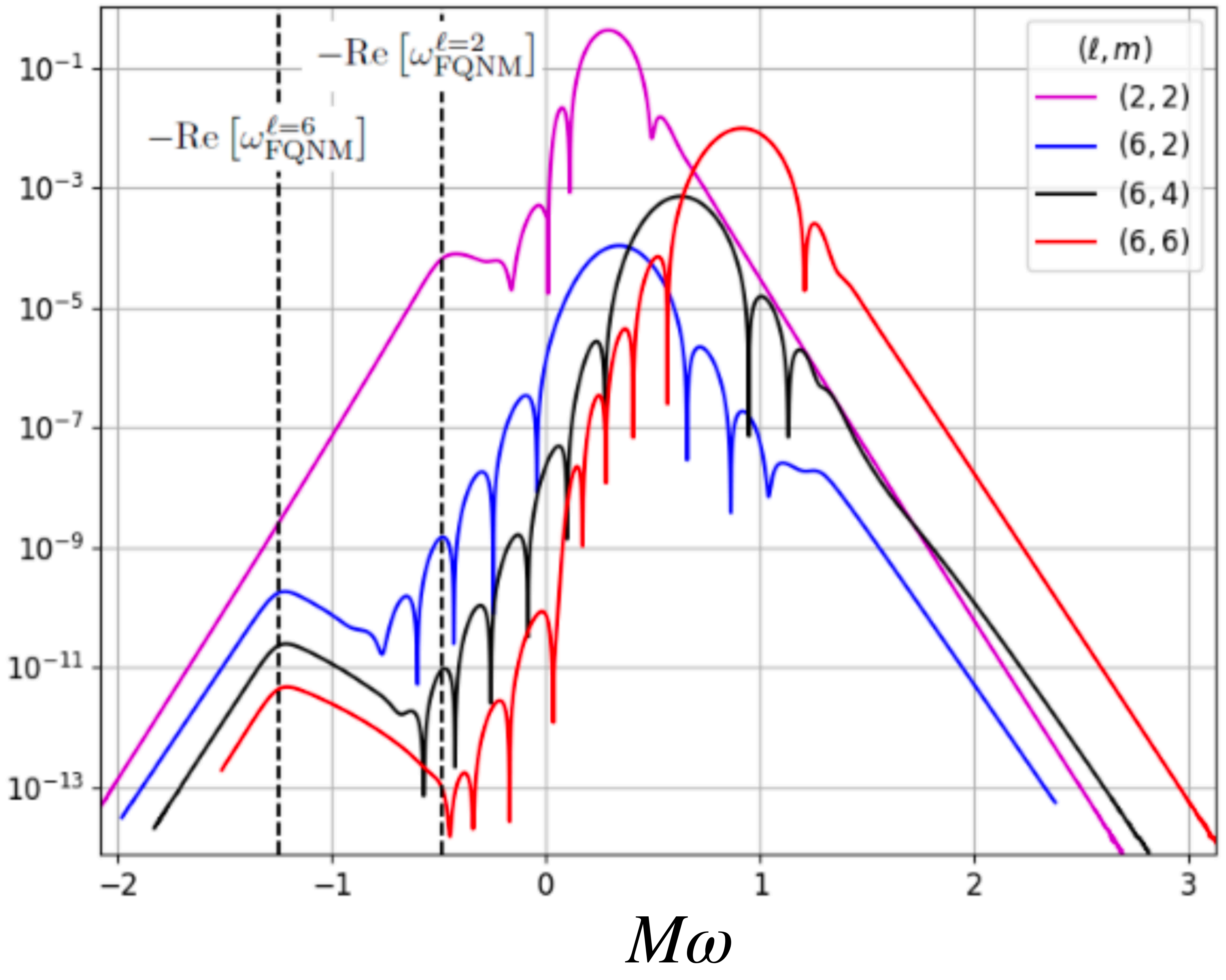
(LB, Bern et al 2023)



f-domain numerical method

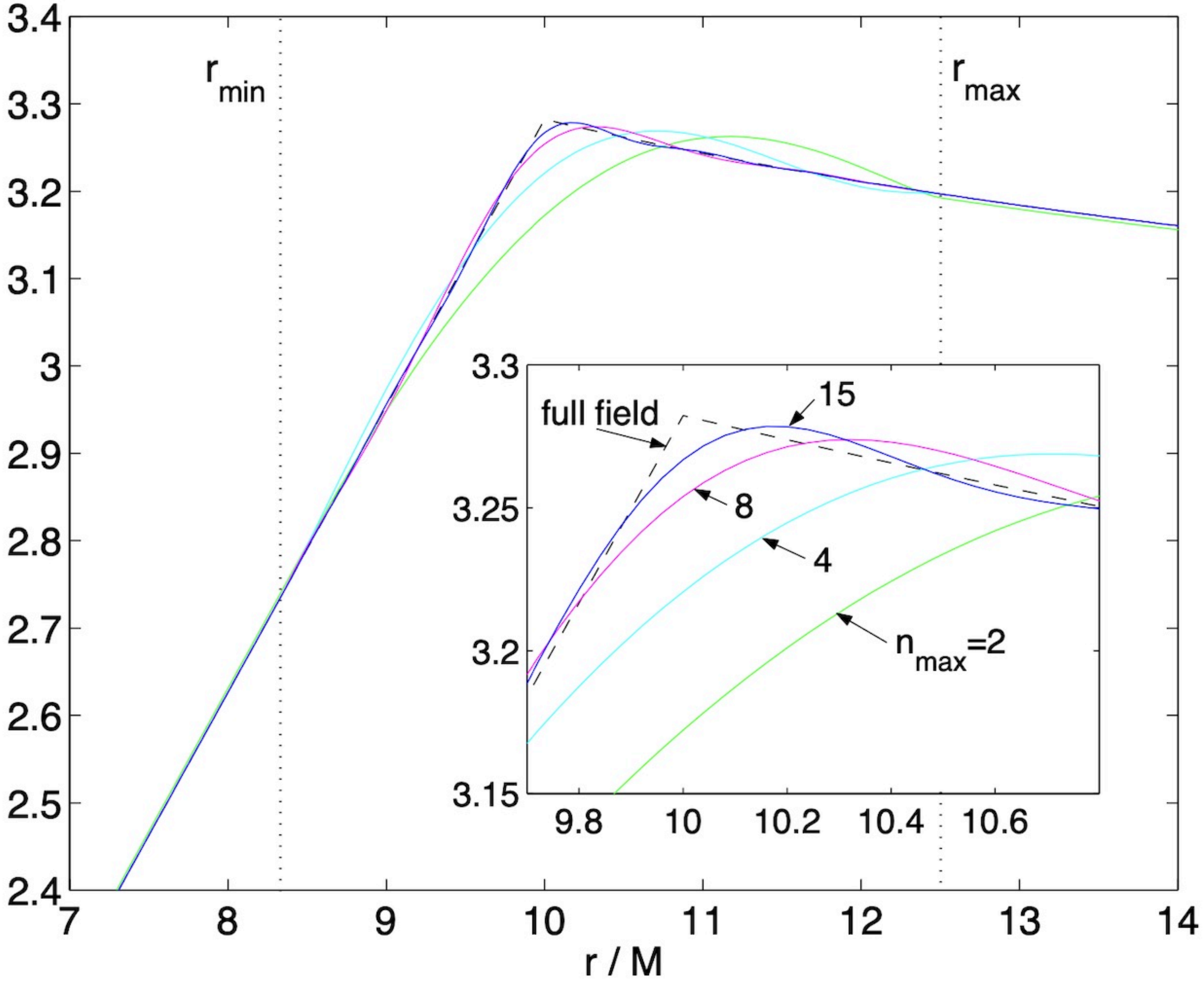
(Whittall & LB 2023)

- $\Phi = \frac{Q}{r}\sum_{\ell,m}\int d\omega\, \phi_{lm\omega}(r) Y_{lm}(\theta,\varphi) e^{-i\omega t}$
- $\phi_{lm\omega}(r)$ obtained by solving ODEs with BCs.
- **Much** more precise than TD method in strong field.
- $\phi_{lm}(r,t)$ reconstructed using the method of extended homogeneous solutions

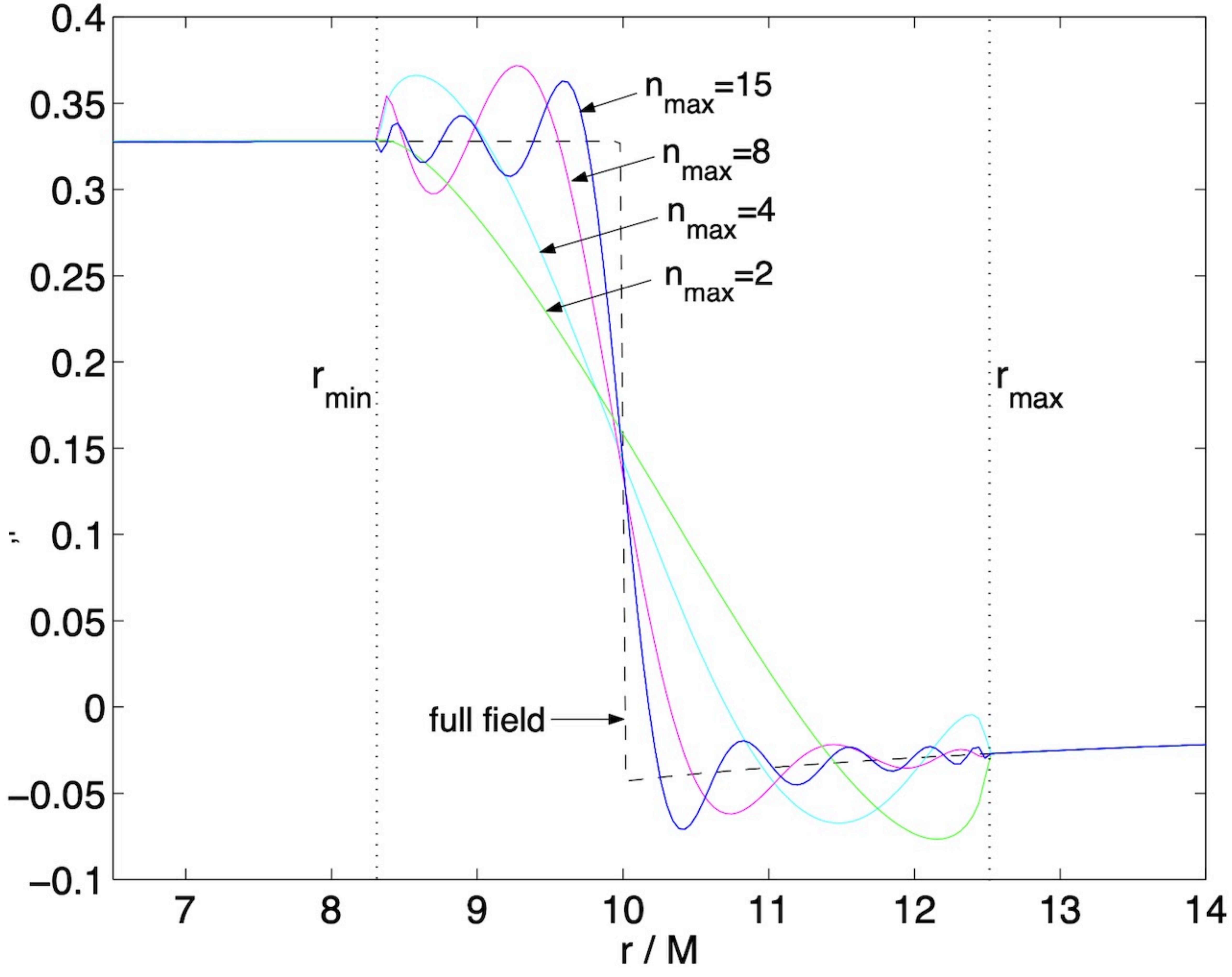


A complication: Gibbs phenomenon

$\phi_{00}(r; t = \text{const})$



$\nabla_r \phi_{00}(r; t = \text{const})$

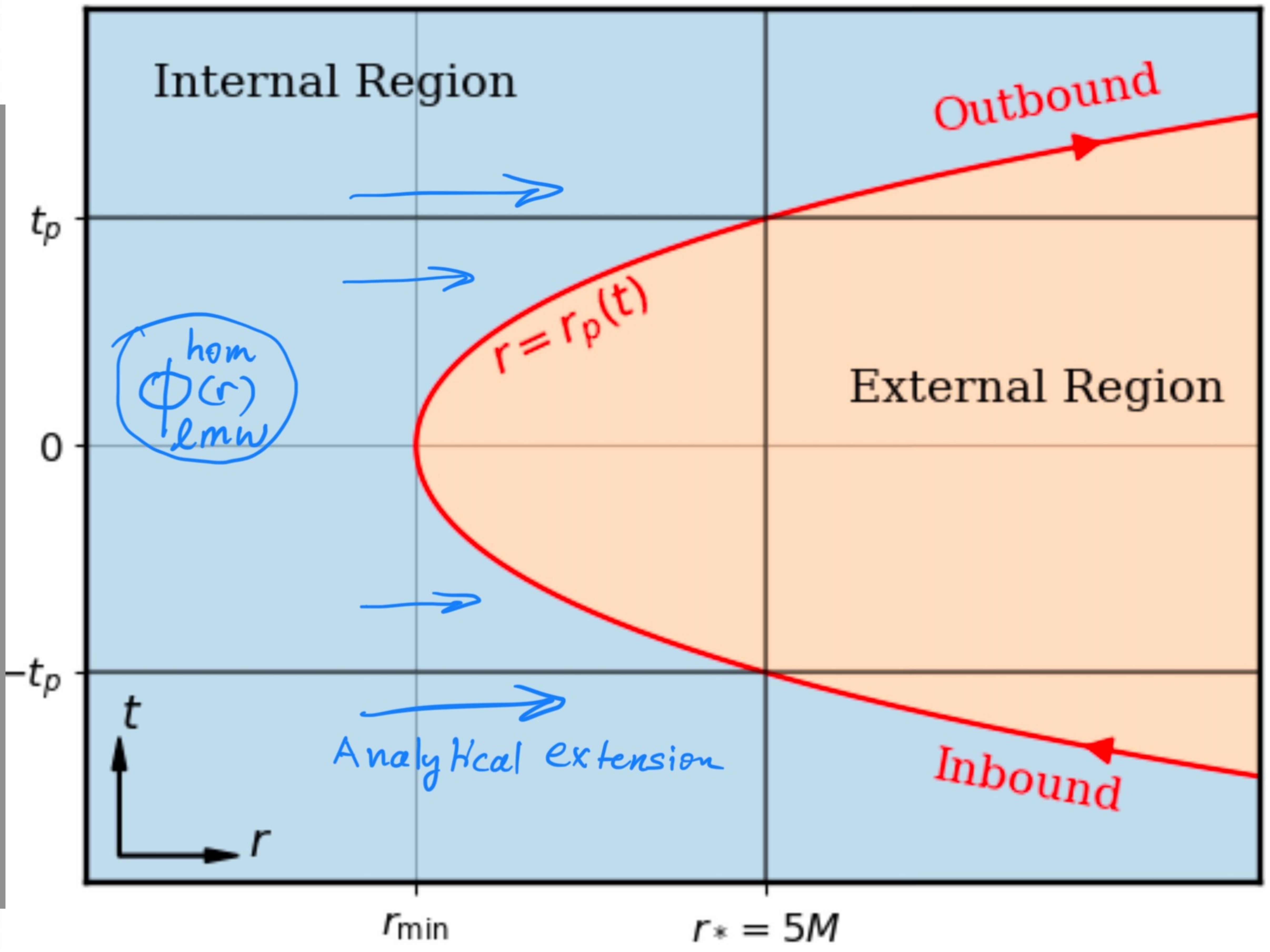


Method of extended homogeneous solutions

(LB, Ori & Sago 2008)

- Trick recovers **exp convergence** of Fourier integral
- However, in scattering problem can only be applied in "internal" region
- Loss of accuracy at large r, especially at large l, due to strong ω-mode cancellation in Fourier integral:

$$\phi^{\rm hom}_{lm\omega} \sim r^l \text{ at small } |\omega|$$

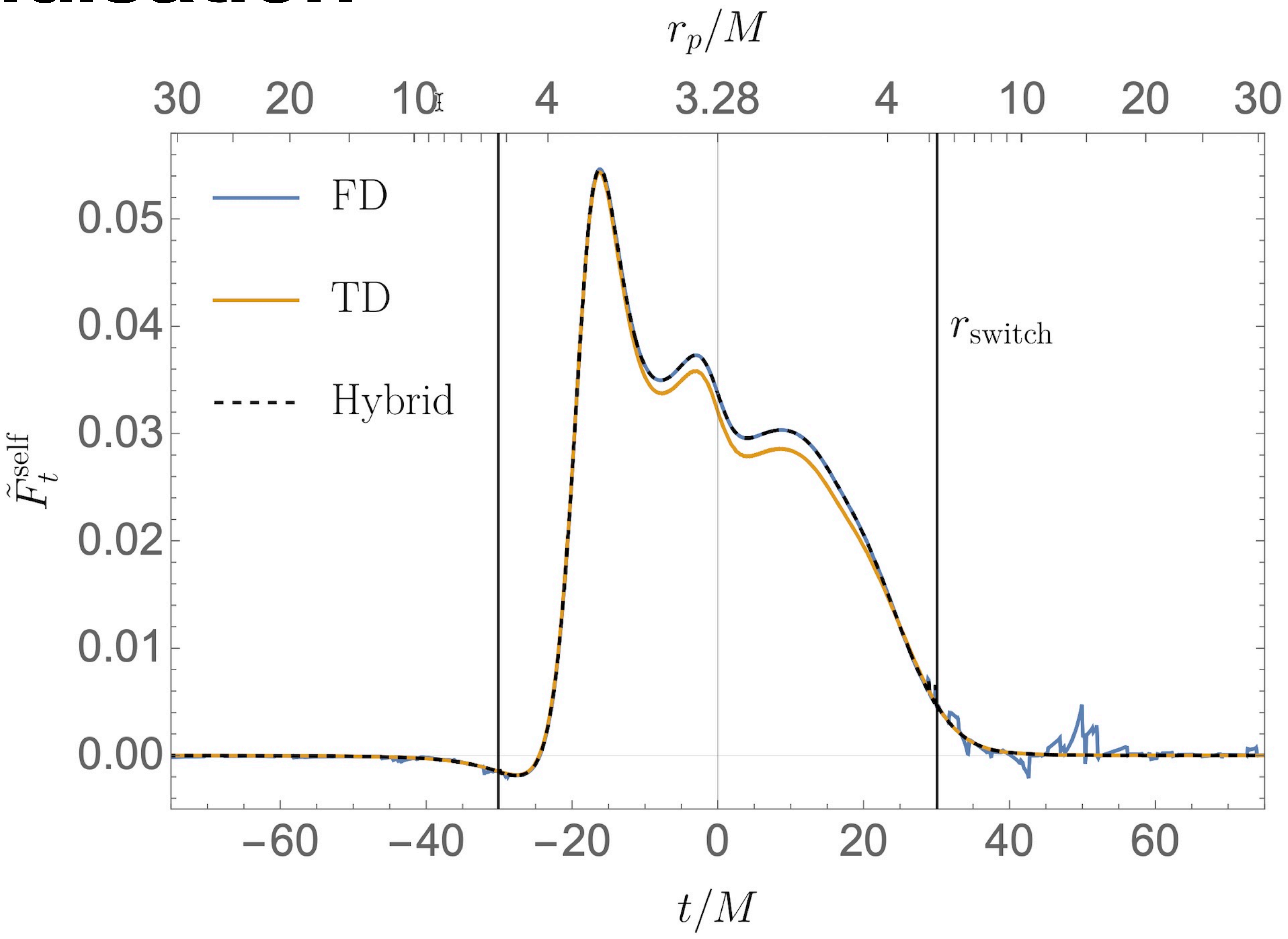


f/t-domain hybridisation

(Long, Whittall & LB 2024)

$v = 0.7 \qquad b - b_{crit} = 0.001M$

(In this case, strong beaming sends power to large l modes, where TD method struggles.)



Application: resummation of χ_{PM} using separatrix info

(Long, Whittall & LB 2024)

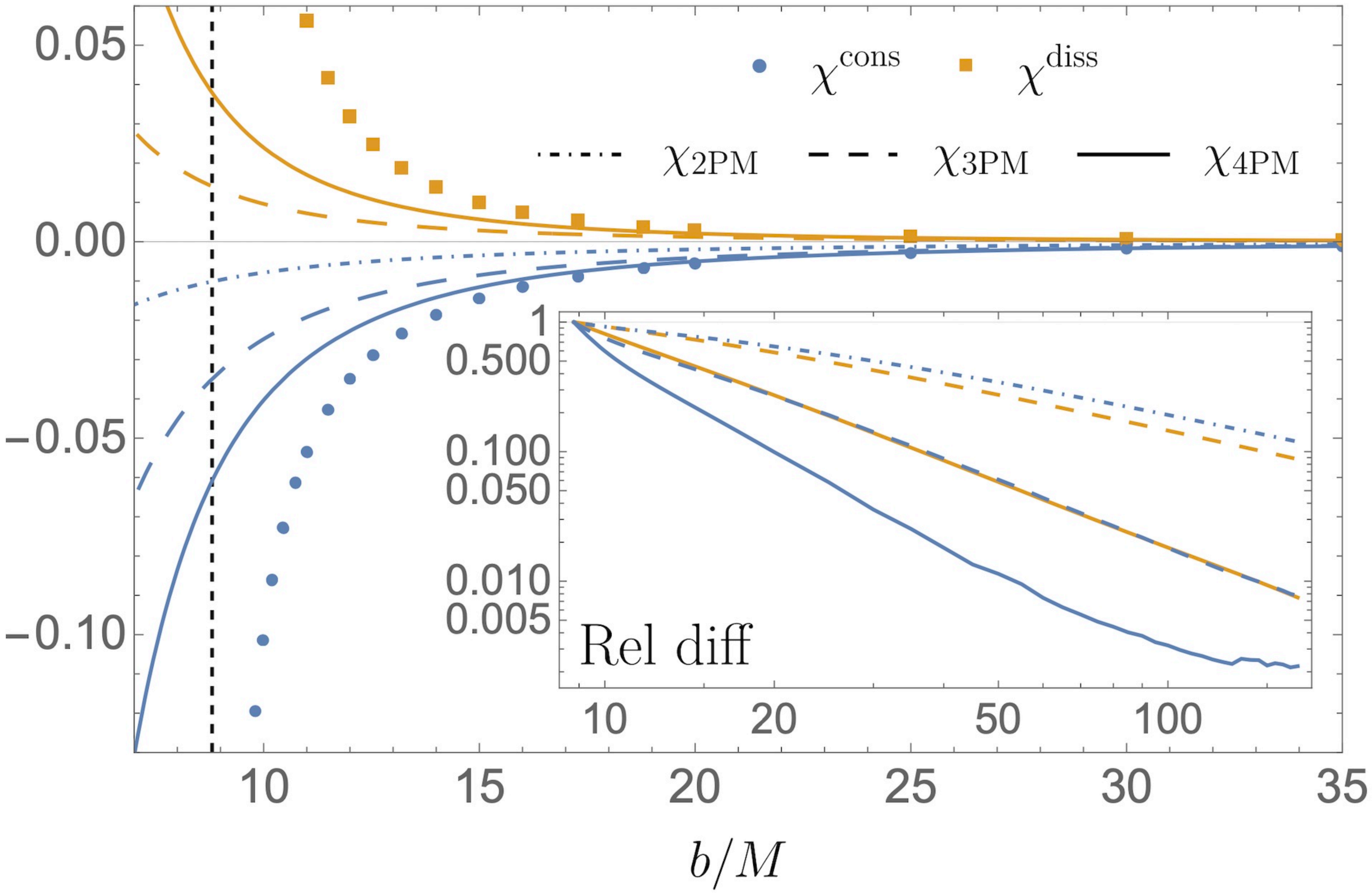
Scattering angle diverges at separatrix:

$$\chi_{0SF} \sim A_0(v)\, \log\big(b - b_{\mathrm{crit}}(v)\big)$$

$$\chi_{1SF} \sim \frac{A_1(v)}{b - b_{\mathrm{crit}}(v)}$$

$$A_1 = \int_{-\infty}^{\infty} \left(c_t(v) F_t^{\mathrm{self}} + c_\varphi(v) F_\varphi^{\mathrm{self}}\right) d\tau$$

$v = 0.5$

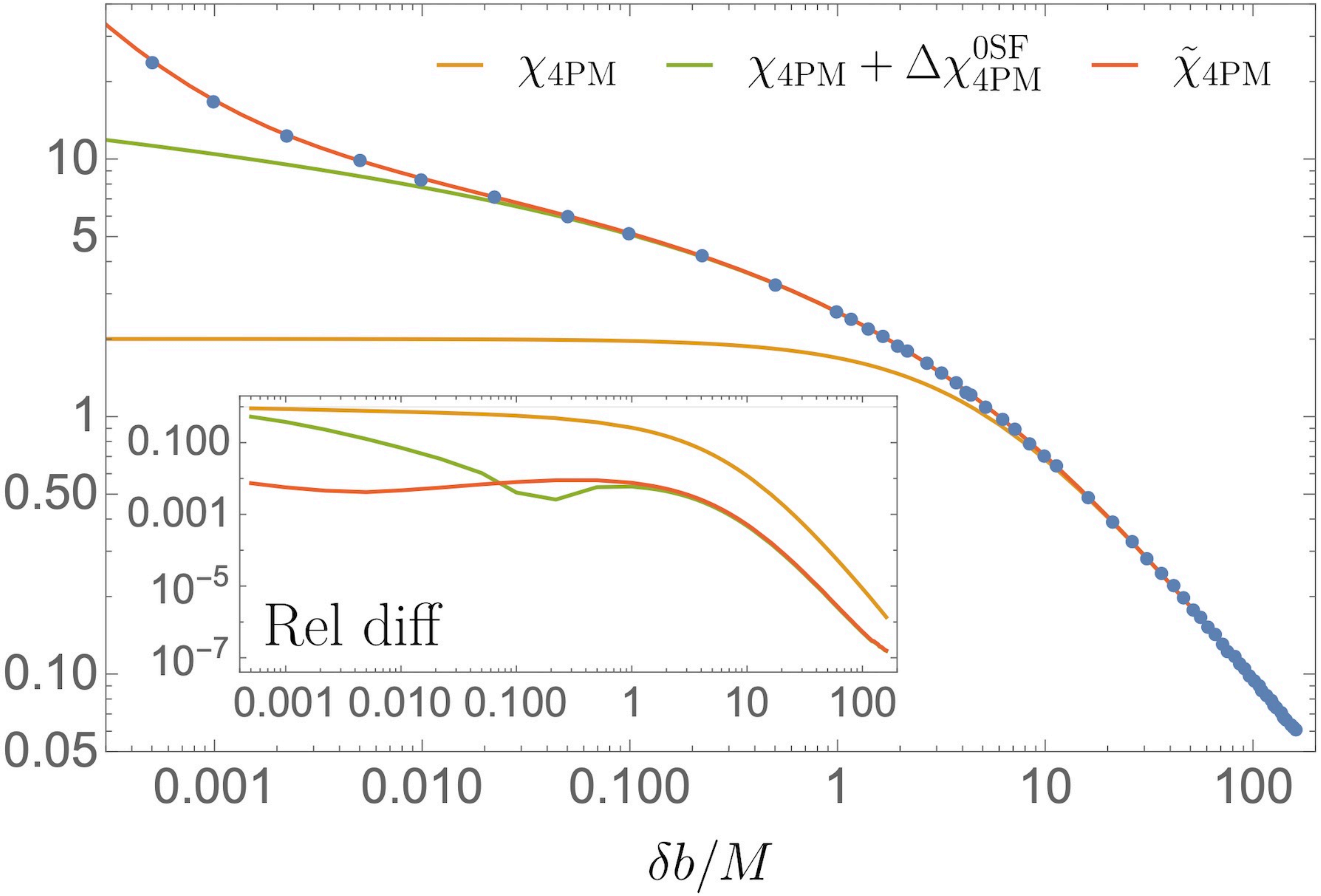


Application: resummation of $\chi_{\rm PM}$ using separatrix info

(Long, Whittall & LB 2024)

Resummation formula:

$$\tilde{\chi} = \chi_{4PM} + A_0 \log\left(1 - \frac{1 - \epsilon A_1/A_0}{b/b_{\rm crit}}\right) + \sum_{k=1}^{4} \frac{A_0}{k}\left(\frac{1 - \epsilon A_1/A_0}{b/b_{\rm crit}}\right)^k$$



Application: resummation of $E_{\rm rad}$ using separatrix info

(LB, Gonzo, Leather, Long & Warburton, in prep 2025)

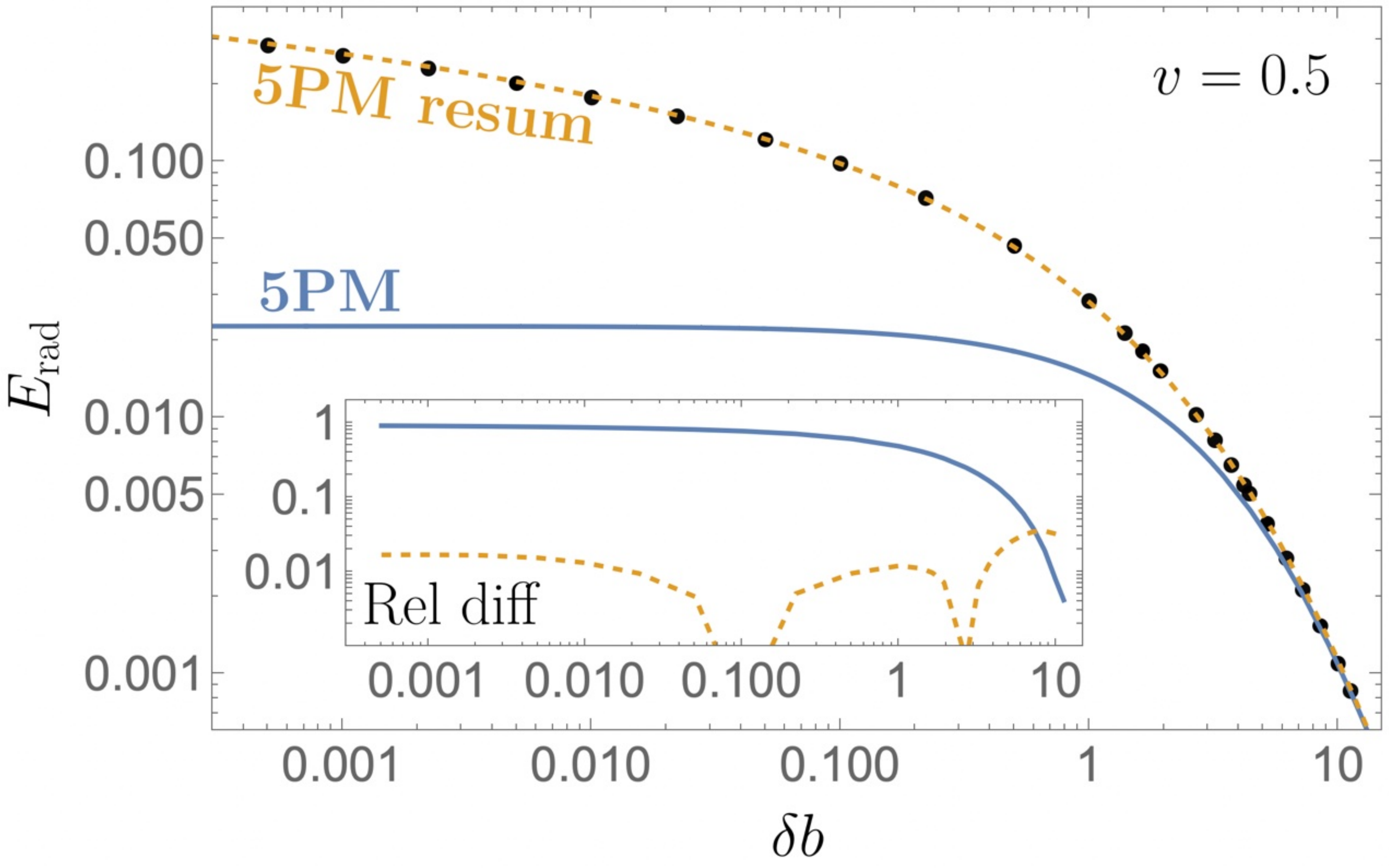
Near separatrix $E_{\rm rad}$ dominated by whirl ⇒

$$E_{\rm rad} \sim \dot{E}_{\rm whirl} \times T_{\rm whirl} \times N_{\rm whirl}$$

$$\sim \dot{E}(R) \times T(R) \times (\chi + \pi)/(2\pi)$$

$$= \dot{E}(R) \times \frac{R^2/M}{\sqrt{6 - R/M}} \log(b - b_{\rm crit})$$

Circular-orbit flux $\dot{E}(R)$ known numerically, with accurate analytical fit over $3M < R \leq 6M$



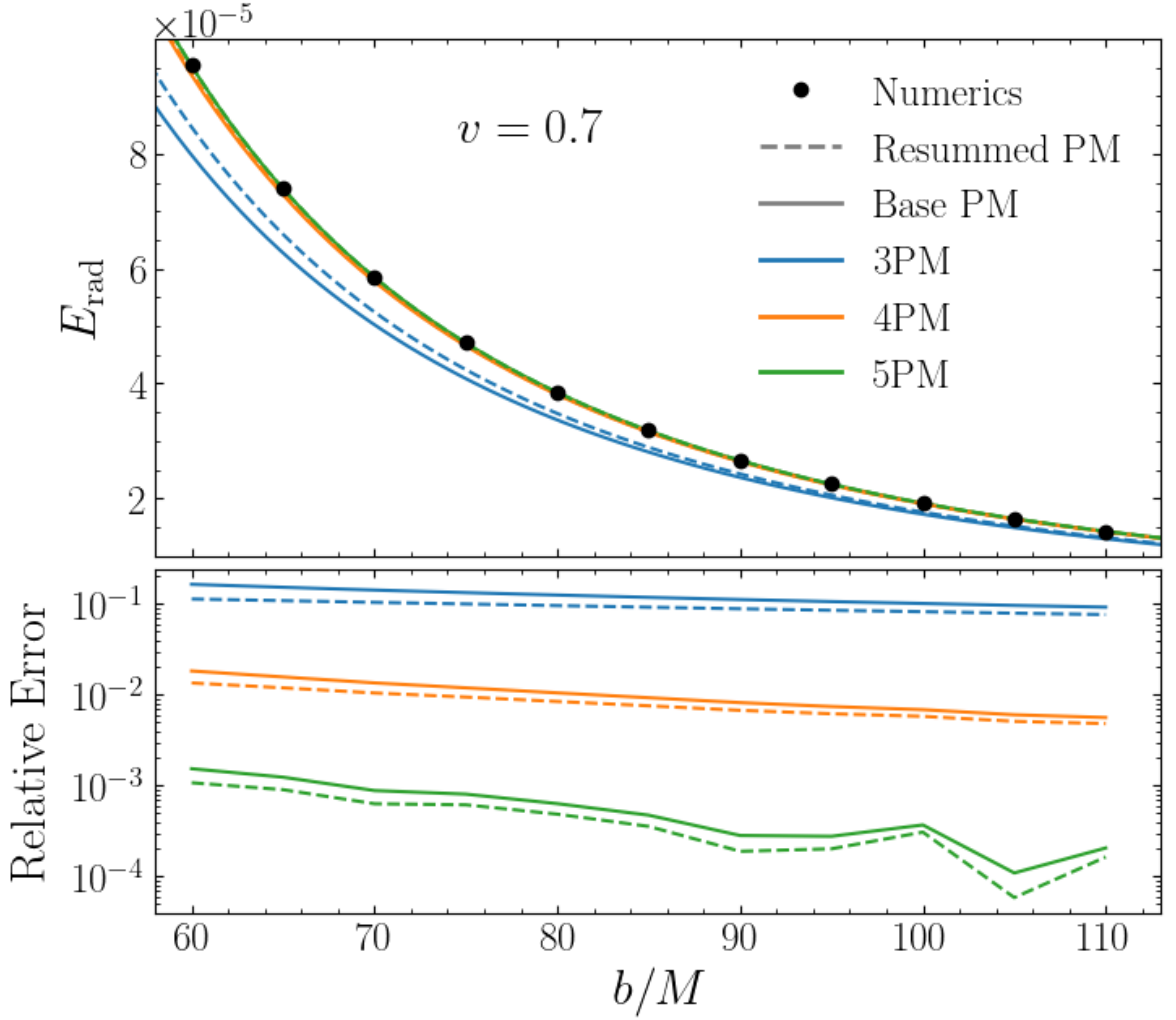
Resummed $E_{\rm rad}$ using separatrix info (grav case)

(LB, Gonzo, Leather, Long & Warburton, in prep 2025)

Near separatrix $E_{\rm rad}$ dominated by whirl $\Rightarrow$

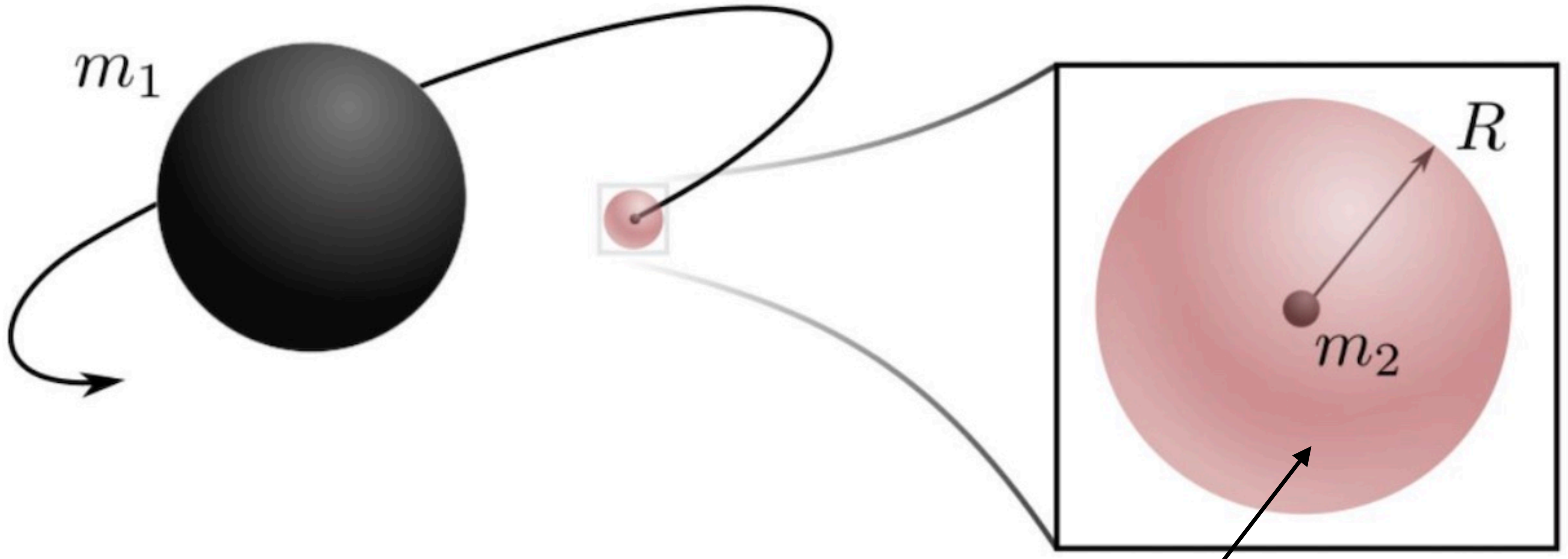
$$E_{\rm rad} \sim \dot{E}_{\rm whirl} \times T_{\rm whirl} \times N_{\rm whirl}$$
$$\sim \dot{E}(R) \times T(R) \times (\chi + \pi)/(2\pi)$$
$$= \dot{E}(R) \times \frac{R^2/M}{\sqrt{6 - R/M}} \log(b - b_{\rm crit})$$

Circular-orbit flux $\dot{E}(R)$ known numerically, with accurate analytical fit over $3M < R \leq 6M$

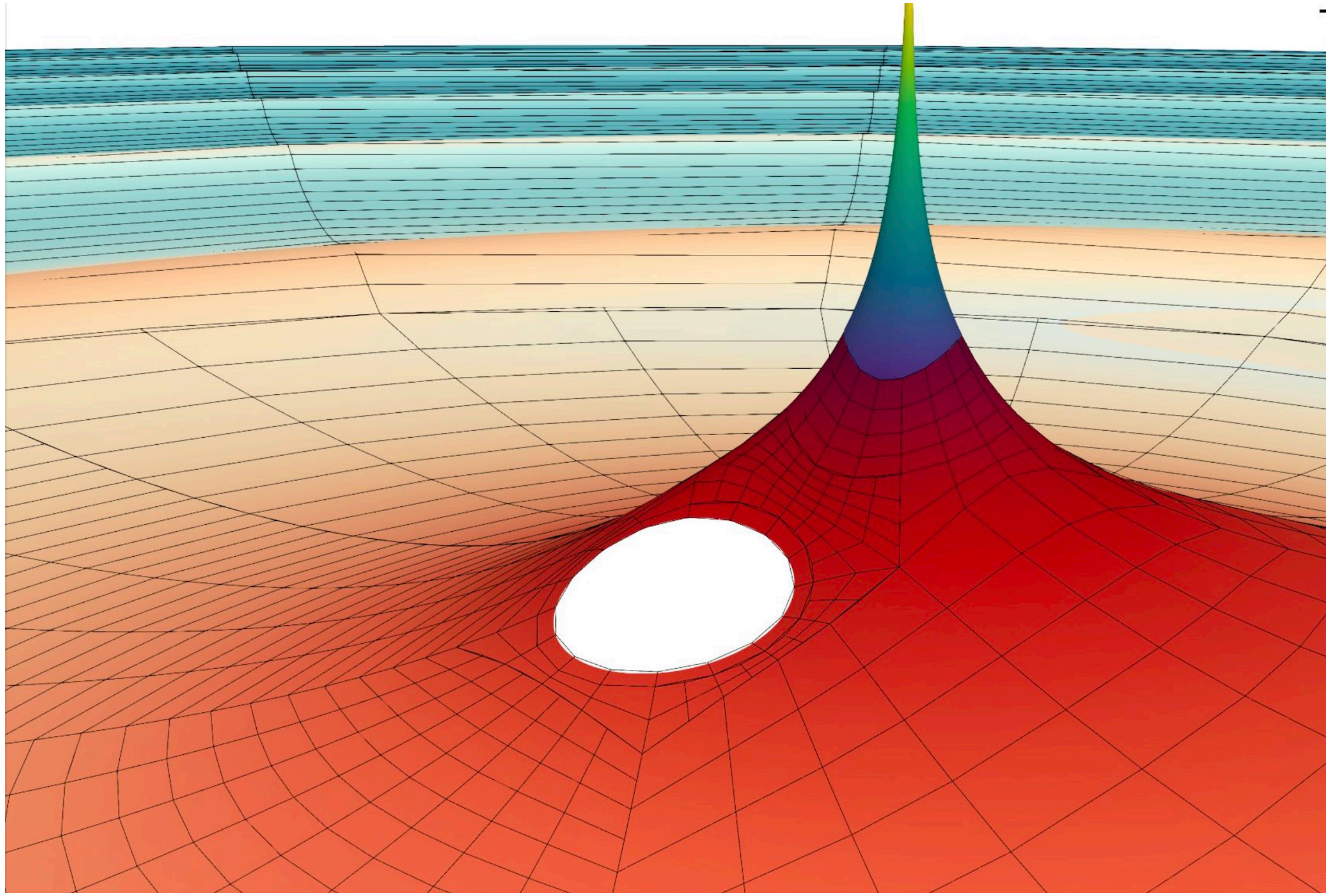


Scattering using NR with worldtube excision

(Wittek, LB, Pfeiffer, Pound + PRL 2025)

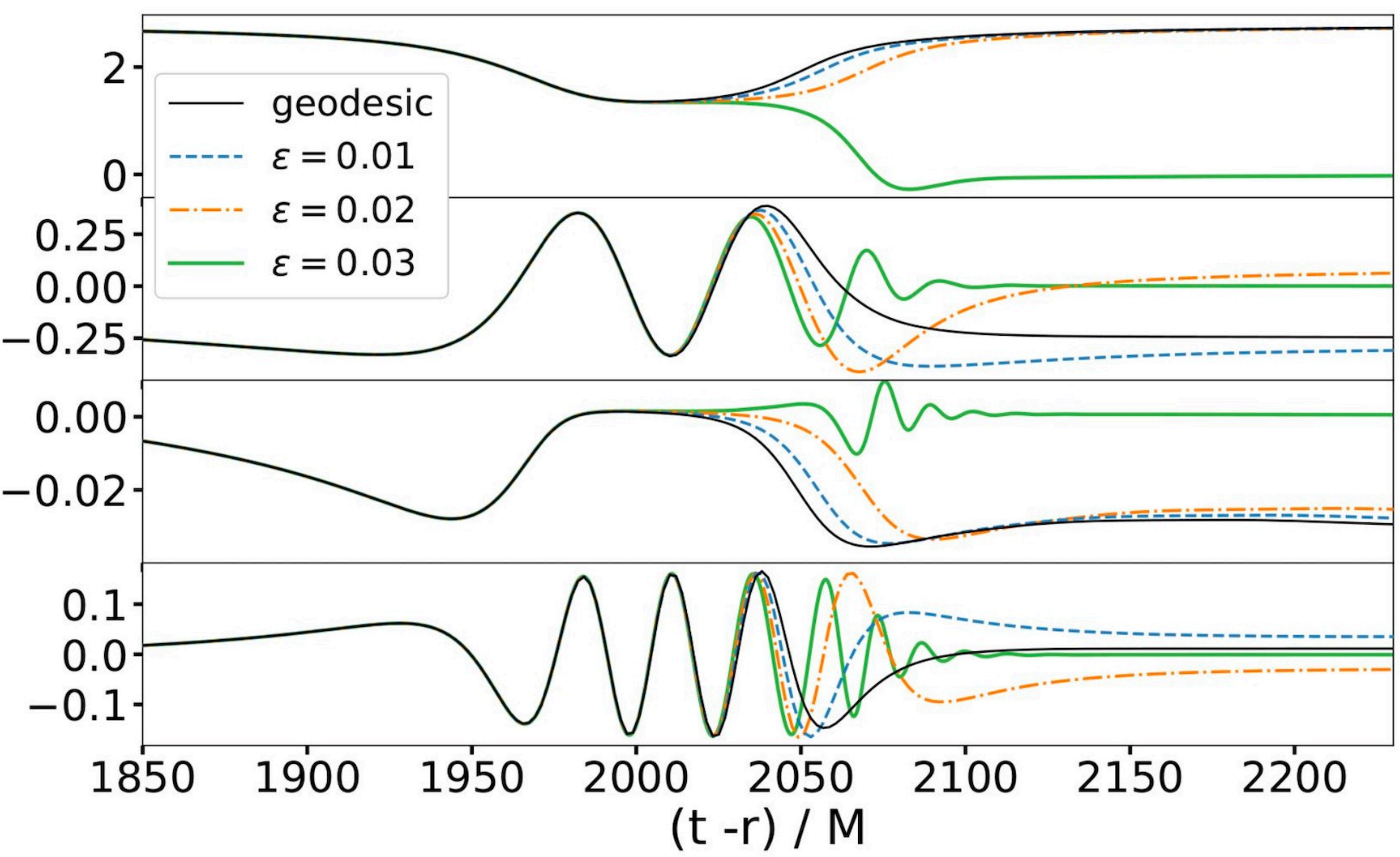


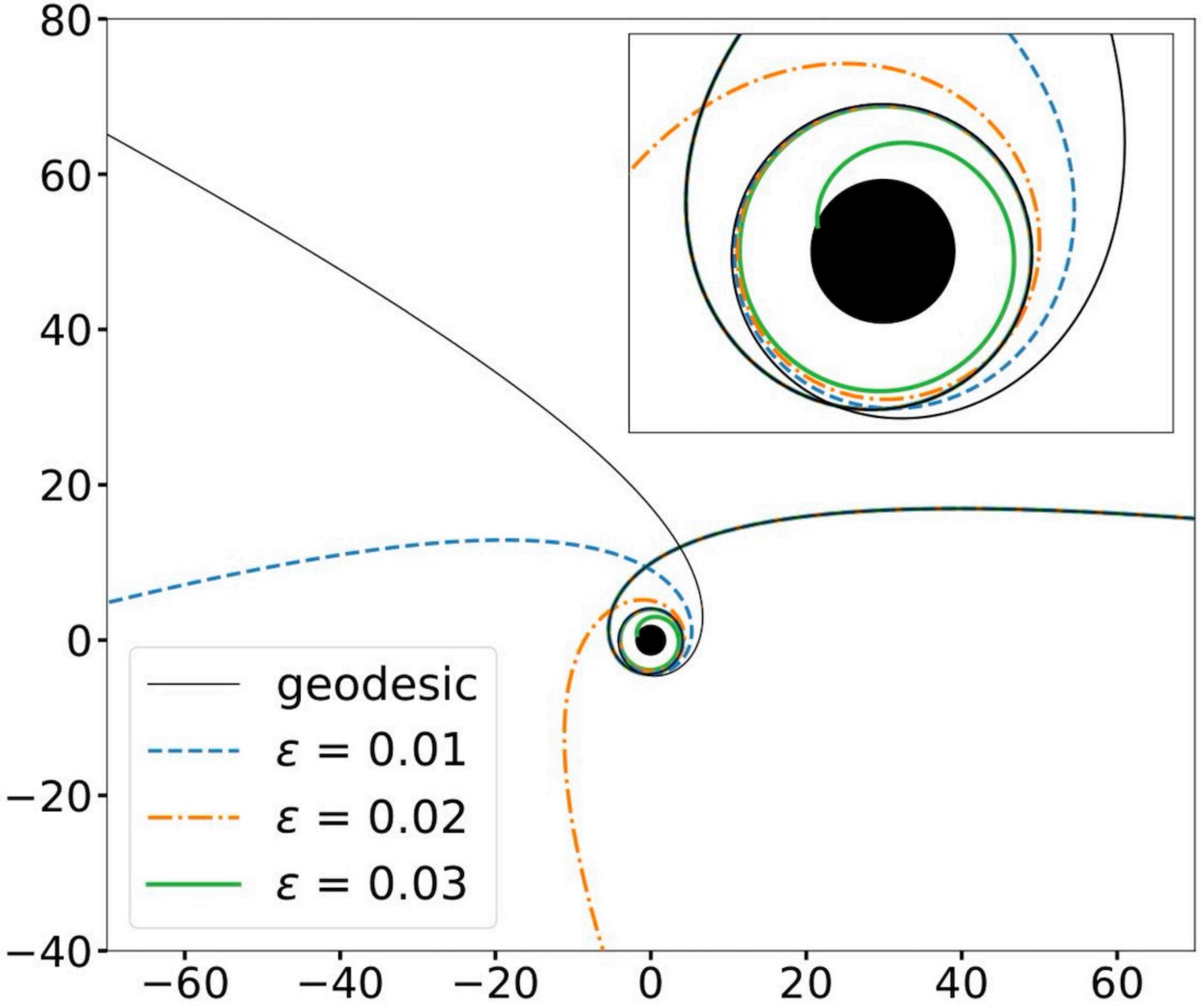
Metric in excision region around scalar charge approximated using self-force theory



Scattering using NR with worldtube excision

(Wittek, LB, Pfeiffer, Pound + PRL 2025)

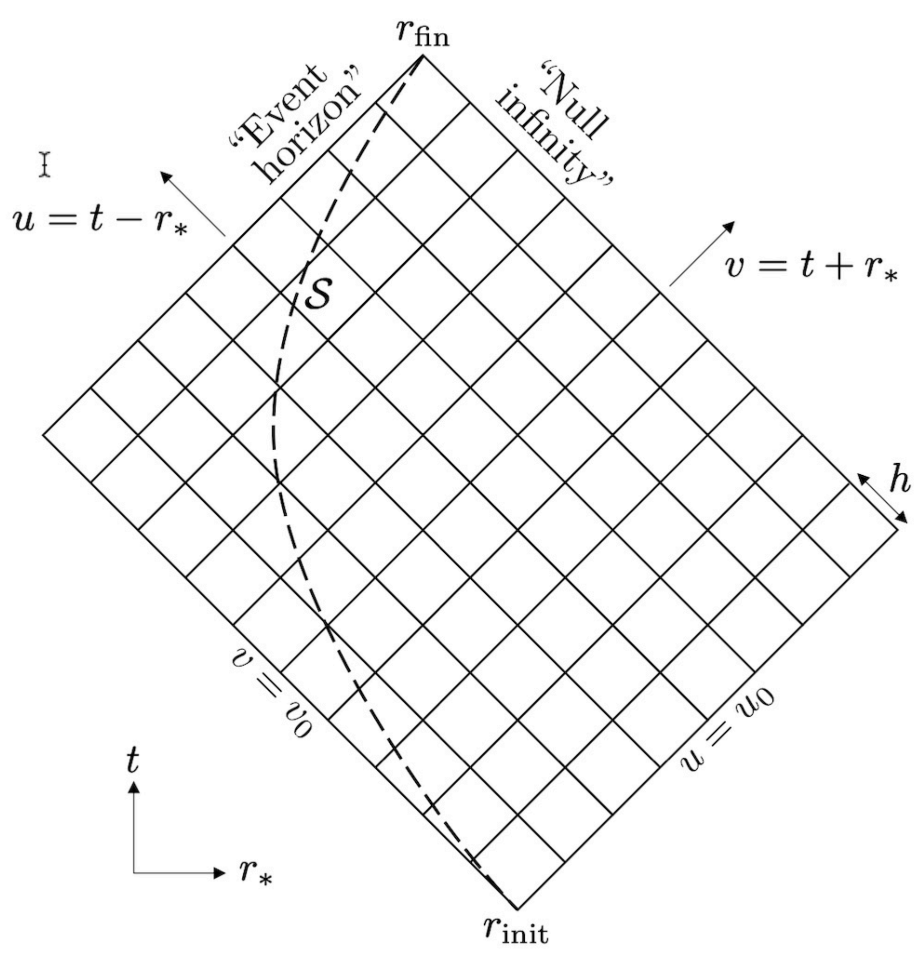


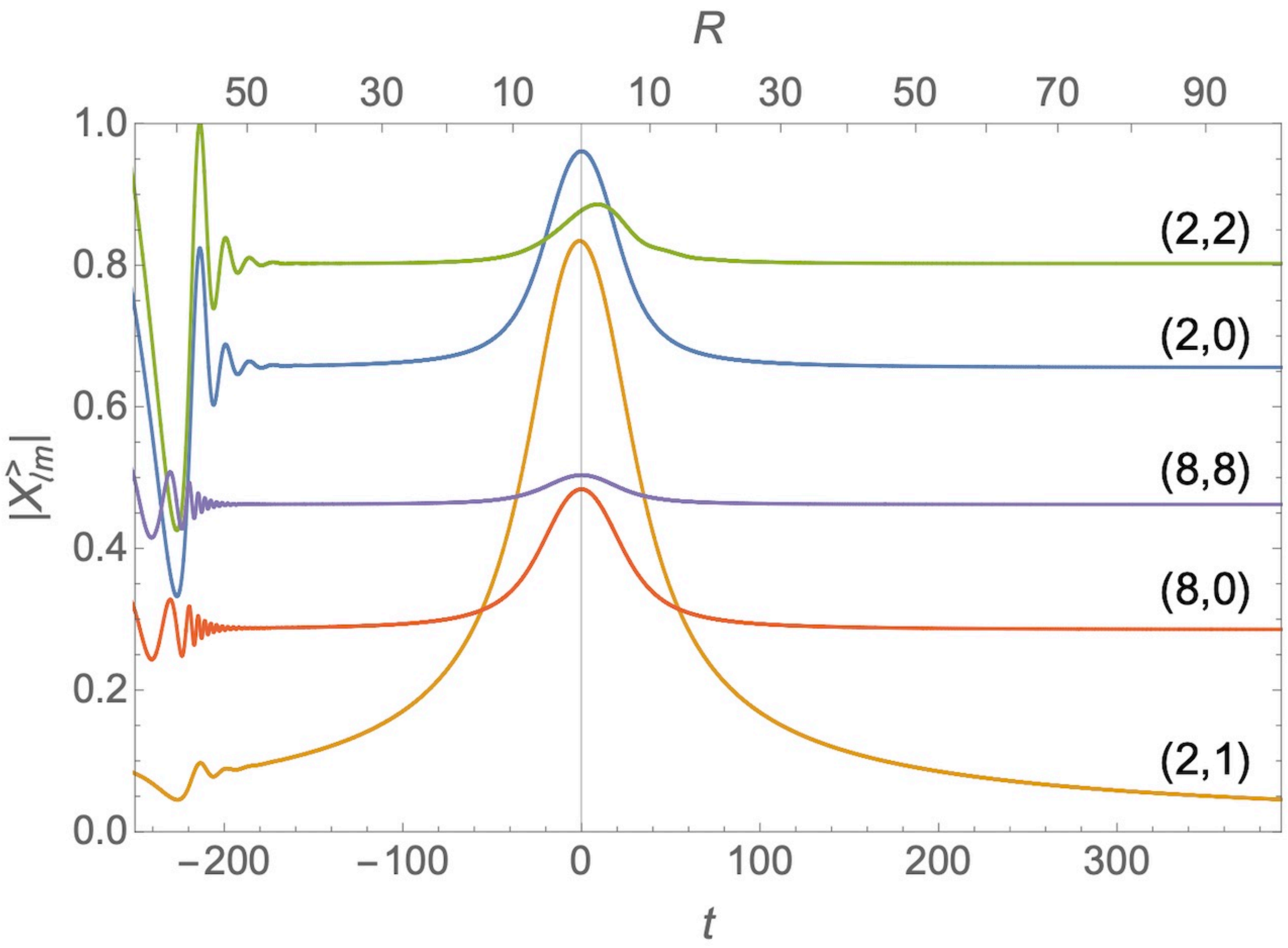


Gravitational scattering: first attempts

- Flux calculations (without metric reconstruction or self-force): Hopper and Cardoso 2018, Warburton 2025, using frequency-domain solutions of the Regge-Wheeler equations.

- LB & Long 2021: metric reconstruction from RW variables using a double-null time-domain code.

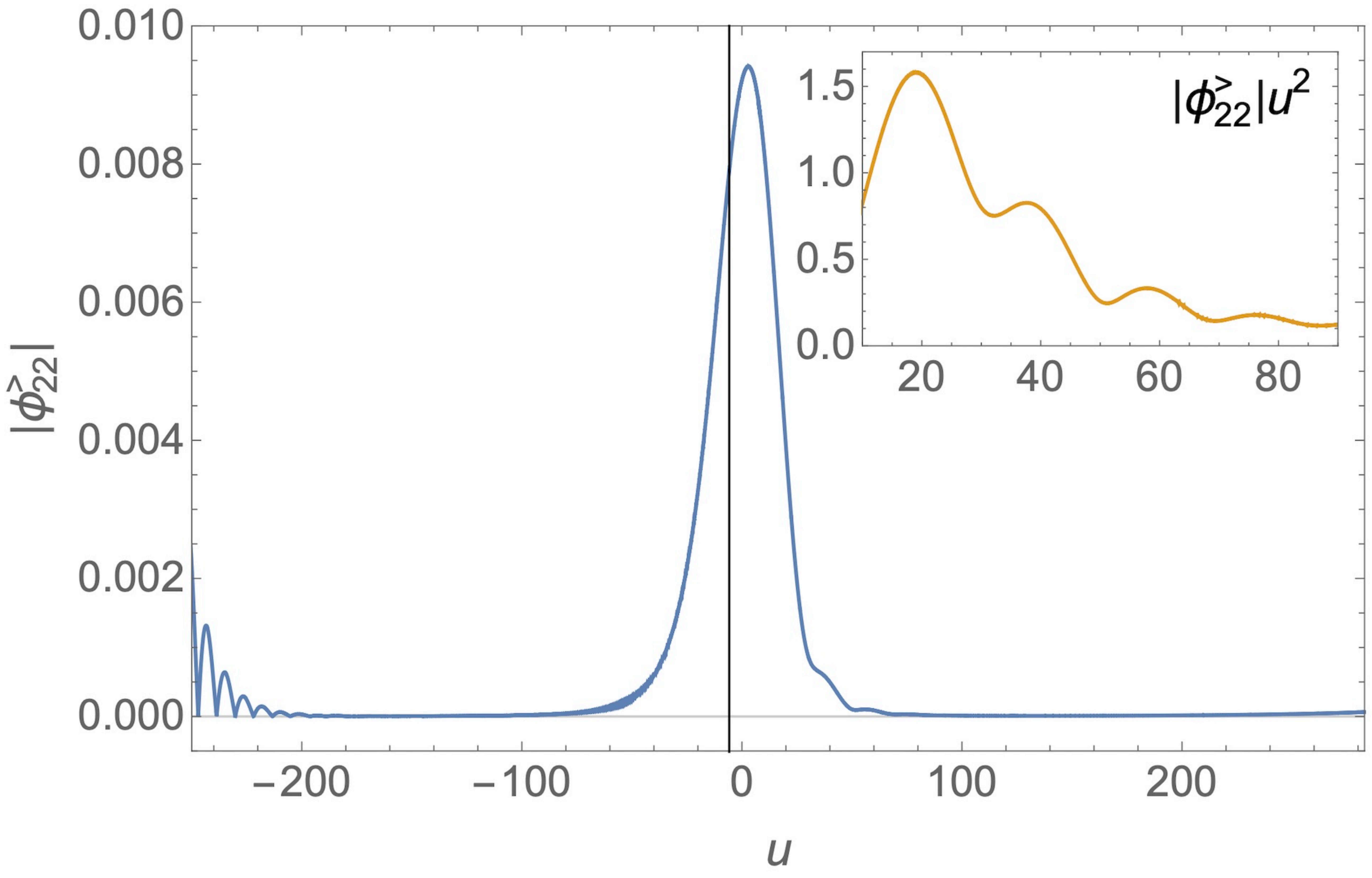




Gravitational scattering: first attempts

(Long & LB, 2021)

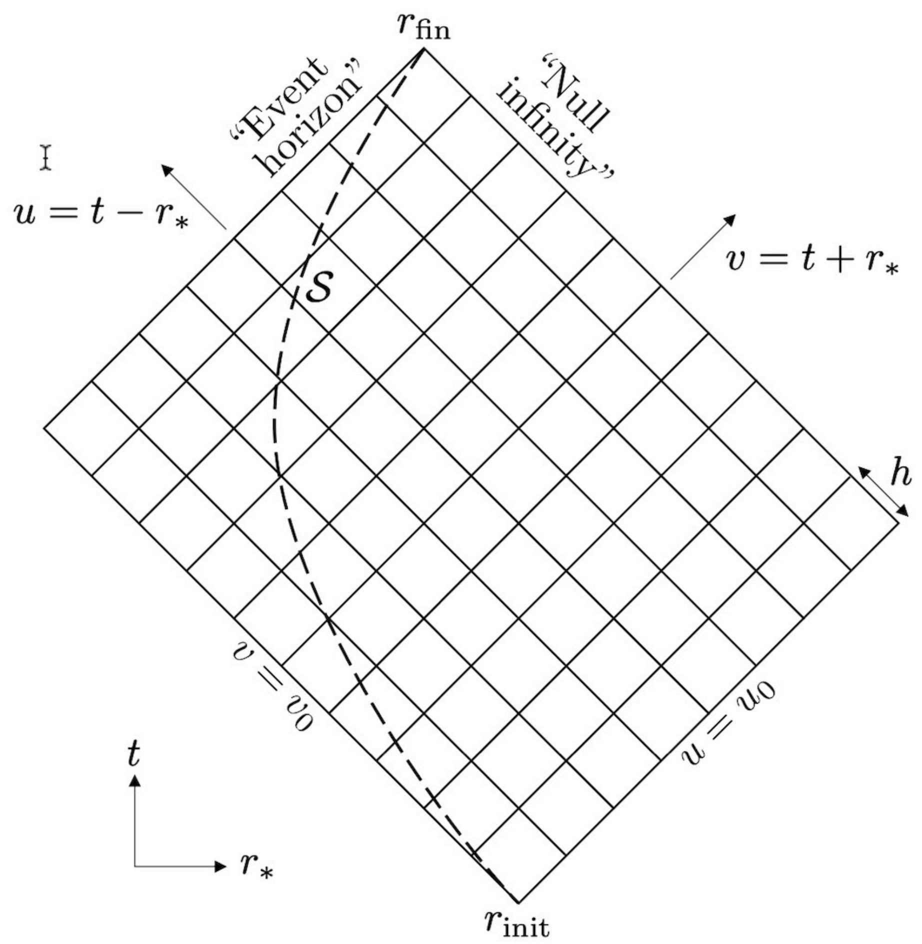
- RW metric not useful for self-force calculations. Instead, apply Chandrasekhar trans. to $s = -2$ **Teukolsky Hertz potential**, from which metric can be reconstructed in radiation gauge - suitable for self-force.
- Alas, procedure involves taking 5th(!) numerical derivative of numerical field, impractical.
- Much preferred: direct integration of the Teukolsky Hertz potential.

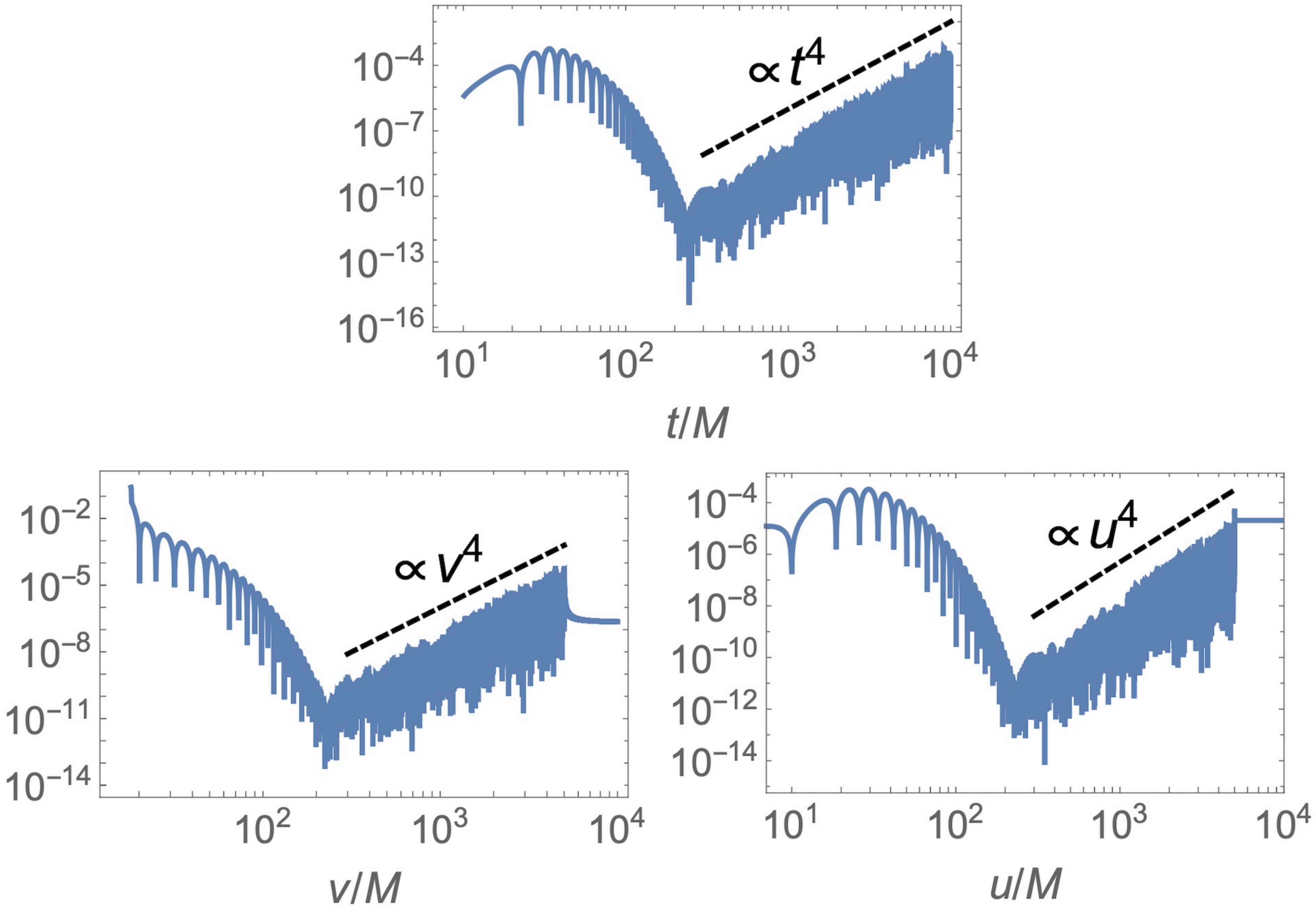


Gravitational scattering: first attempts

(Long & LB, 2021)

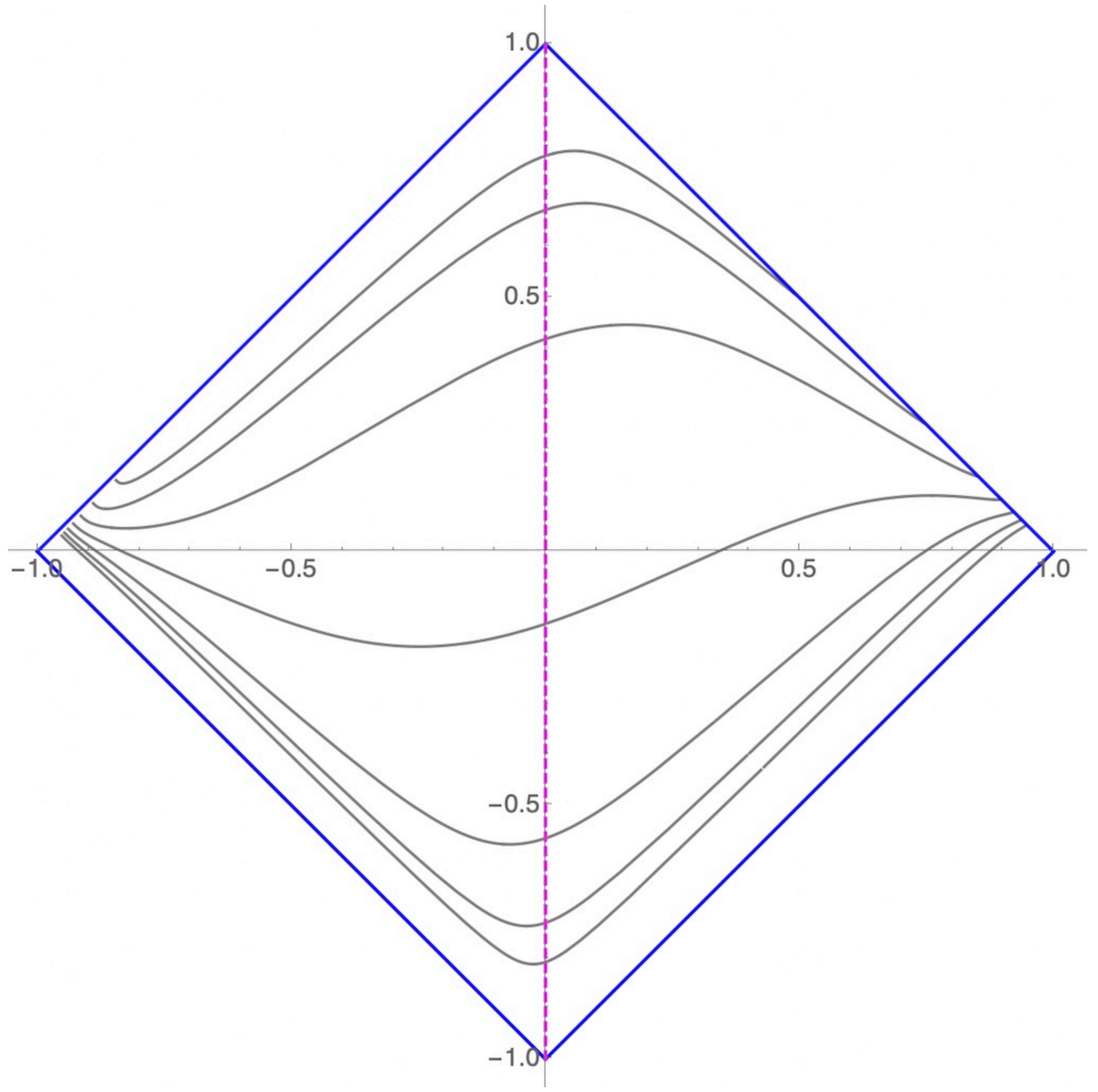
- $s = -2$ Teukolsky equation develops $\sim t^4$ divergence at late time.
- This is due to contamination from “advanced” modes, uncontrollable in our scheme.
- $s = +2$ case is even worse: $\sim e^{t/(2M)}$ divergence

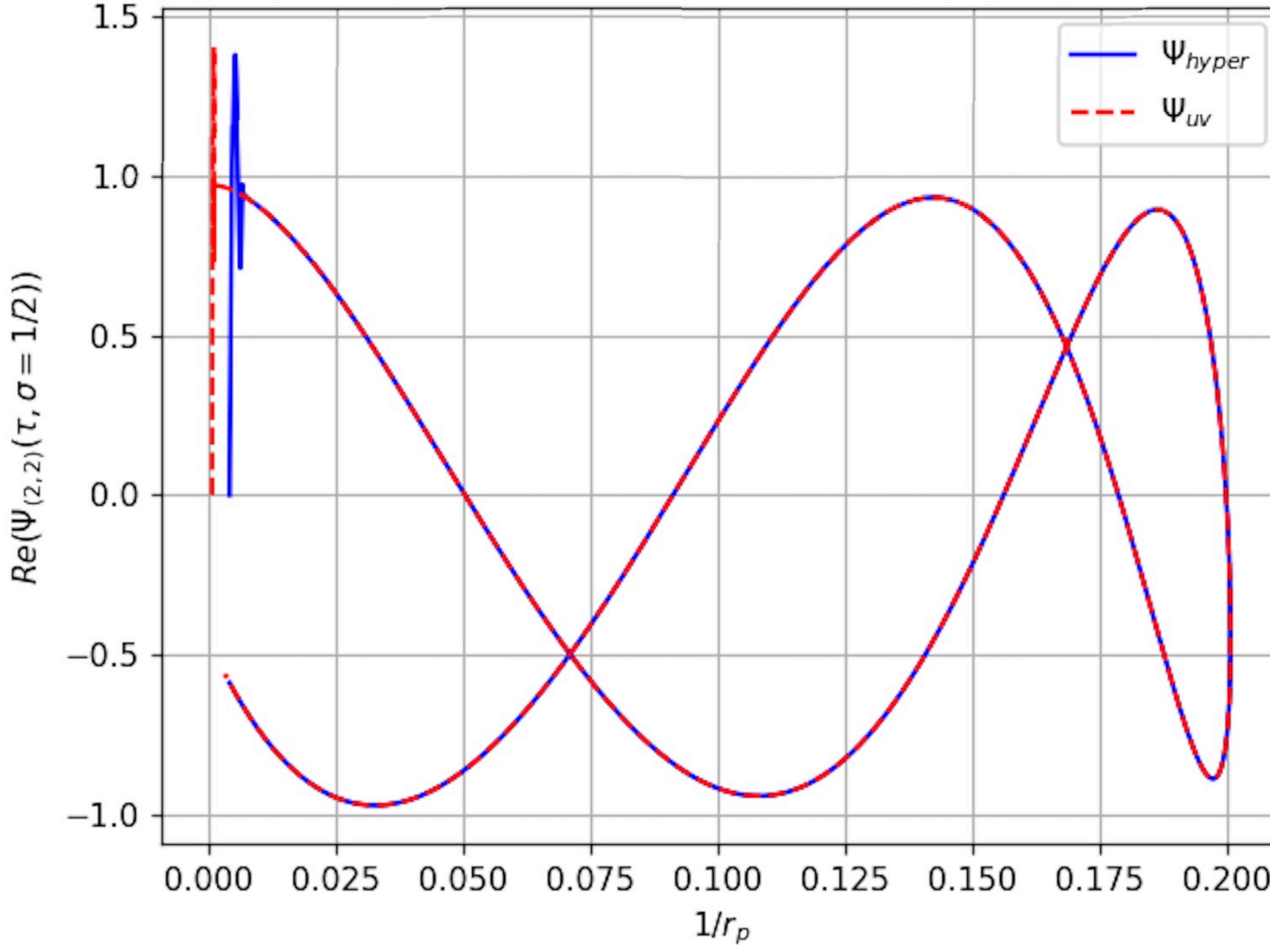




Double the hype: hyperbolic scattering using comoving hyperboloidal coords (Vaswani & LB; Macedo, LB, Long & Vaswani, in prep 2025)

- Compactification should ensure no contamination from advanced modes.
- So far tried only with a scalar field
- Two versions: finite-difference and full spectral





Fresh ideas: Gegenbauer reconstruction

(Whittall, LB and Long, in prep 2025)

- A method for overcoming difficulties inherent to Extended Homogeneous Solutions in f-domain calculations

- Restores exp convergence at particle without mode cancellation problem. Also allows reconstruction of field in the "exterior" region, where EHS fails.

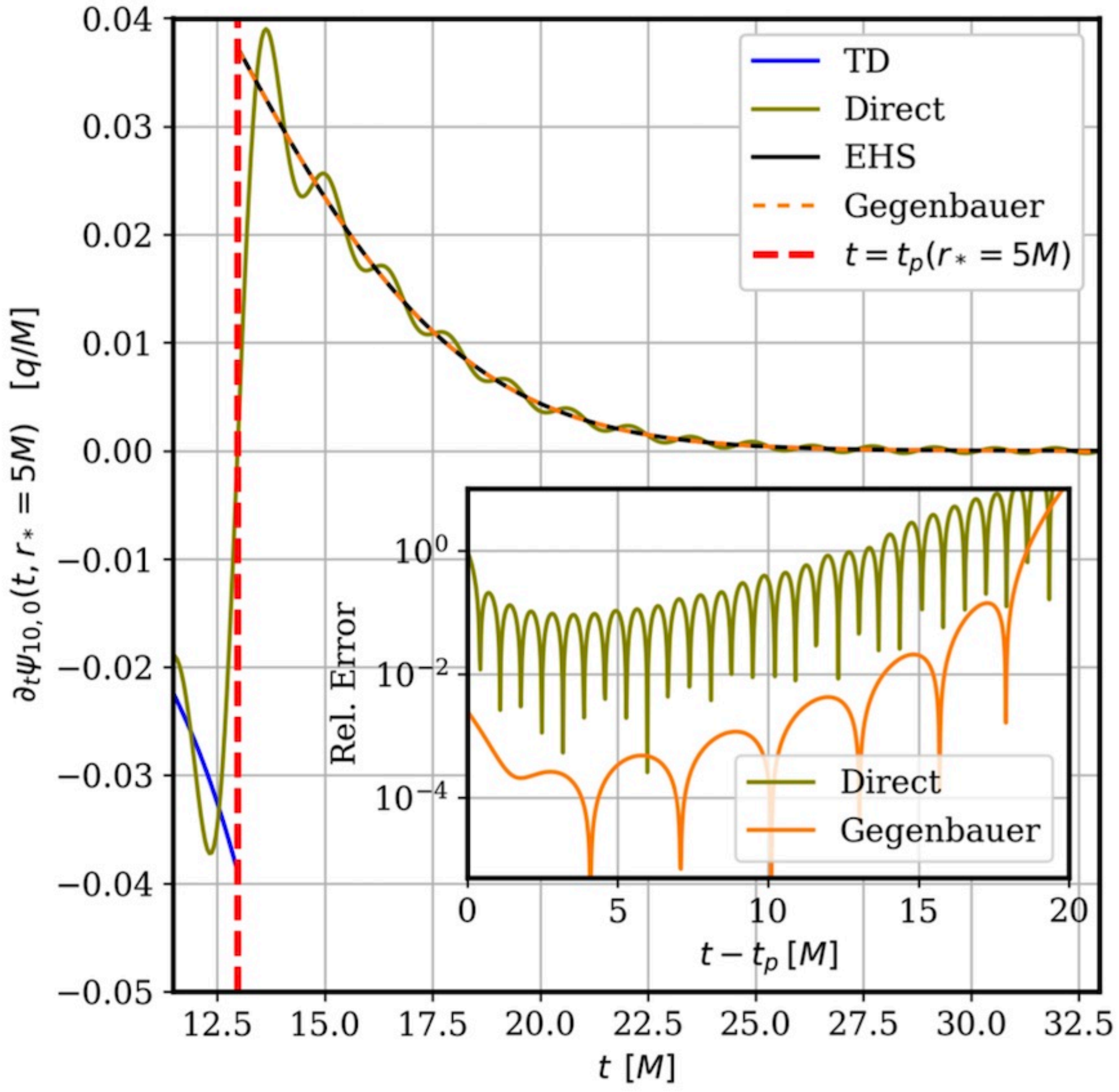
- **Procedure:**

  Re-expand partial Fourier integral in Gegenbauer polynomials:

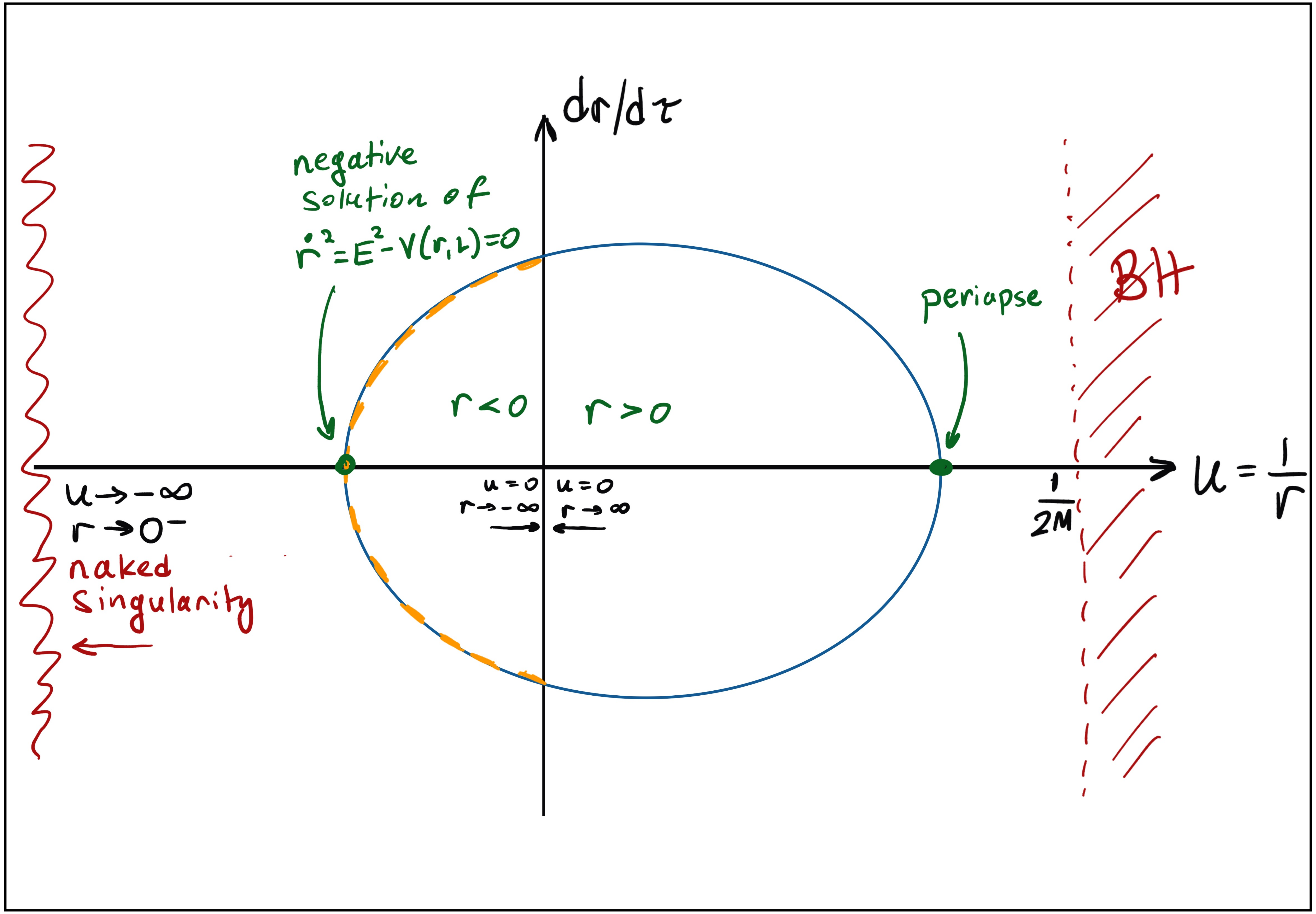
$$F(t;\omega_{\max}) := \int_{-\omega_{\max}}^{\omega_{\max}} \hat{f}(\omega)e^{-i\omega t}d\omega = \sum_{k=0}^{\infty} g_k^{\lambda}(\omega_{\max})C_k^{\lambda}(t).$$

Then approximate $F(t)$ with $G_N(t;\lambda,\omega_{\max}) := \sum_{k=0}^{N} g_k^{\lambda}(\omega_{\max})C_k^{\lambda}(t).$

- Can prove that G_N converges **exponentially fast** to $F(t)$ as $\omega_{\max} \to \infty$ with fixed $N/\omega_{\max}$ and $\lambda/\omega_{\max}$.



Fresh ideas: restoring periodicity with analytic continuation



Prospects

- **Time-domain method**: modern spectral/hyperboloidal code for Teukolsky equation or direct EFE integration in Lorenz gauge.
- **Frequency-domain method**: using Gegenbauer reconstruction and/or analytical extension to exploit periodicity.
- **Analytical self-force** calculation at large r is underway (LB & Whittall, in progress)
- **2nd-order self-force** in scattering: formulation is underway, including frame fixing (Leplat, Pound, LB & Vaswani, in progress)
- Comparisons with **NR simulations**
- **PM resummation** and determination of high-order PM terms

credit: **Olly Long**, visualisation of SXS:BBH:4292

extras

Boundary-to-bound maps

Relations (established using EFT techniques) between bound-orbit & scattering observables, obtained via **analytic continuation** in parameter space:

- Periastron advance from scattering angle (shown in PM, EOB, 0SF):

$$\Delta\phi(E,J) = \chi(E,J) + \chi(E,-J)$$

- Radiative energy & angular momentum loss (PM) (Cho, Kaelin, Porto 2022):

$$\Delta E_{\mathrm{ellip}}(E,J) = \Delta E_{\mathrm{hyp}}(E,J) - \Delta E_{\mathrm{hyp}}(E,-J)$$

- Bound-orbit waveform snapshots from scattering amplitudes (PM) (Adamo, Gonzo, Ilderton 2024)