Generalizations of Galileon Duality

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Summary

* Why modify gravity?
* Uniqueness of GR
* Review of Massive Gravity/Bigravity
* Emergence of Galileons
* Galileon Duality (Old Story)
* Generalized Galileon Duality (New Story)

WHY MODIFY GRAVITY?

Why modify gravity?

Type I: UV Modifications:

eg. Quantum Gravity, string theory, extra dimensions, branes, supergravity

At energies well below the scale of new physics Λ , gravitational effects are well incorporated in the language of Effective Field Theories

$$S = M_{\rm Planck}^2 \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \frac{a}{\Lambda^2} R^2 + \frac{b}{\Lambda^2} R_{\mu\nu}^2 + \cdots + \frac{c}{\Lambda^4} R_{abcd} R^{cd}_{ef} R^{efab} + \cdots + \mathcal{L}_{\rm matter} \right]$$
$$+ \frac{d}{\Lambda^6} (R_{abcd} R^{abcd})^2 + \ldots$$

eg Cardoso et al 2018

Addition of Higher Dimension, (generally higher derivative operators), **no failure of well-posedness/ghosts** etc as all such operators should be treated perturbatively (rules of EFT)

Type 2: IR Modifications:

Principle Motivation is Cosmological:

Dark Energy and Cosmological Constant

I: Old cosmological constant problem:

Why is the universe not accelerating at a gigantic rate determined by the vacuum energy?

II: New cosmological constant problem:

Assuming I is solved, what gives rise to the remaining vacuum energy or dark energy which leads to the acceleration we observe?

Why modify gravity (in the IR)?

III: Because it allows us to put better constraints on Einstein gravity!

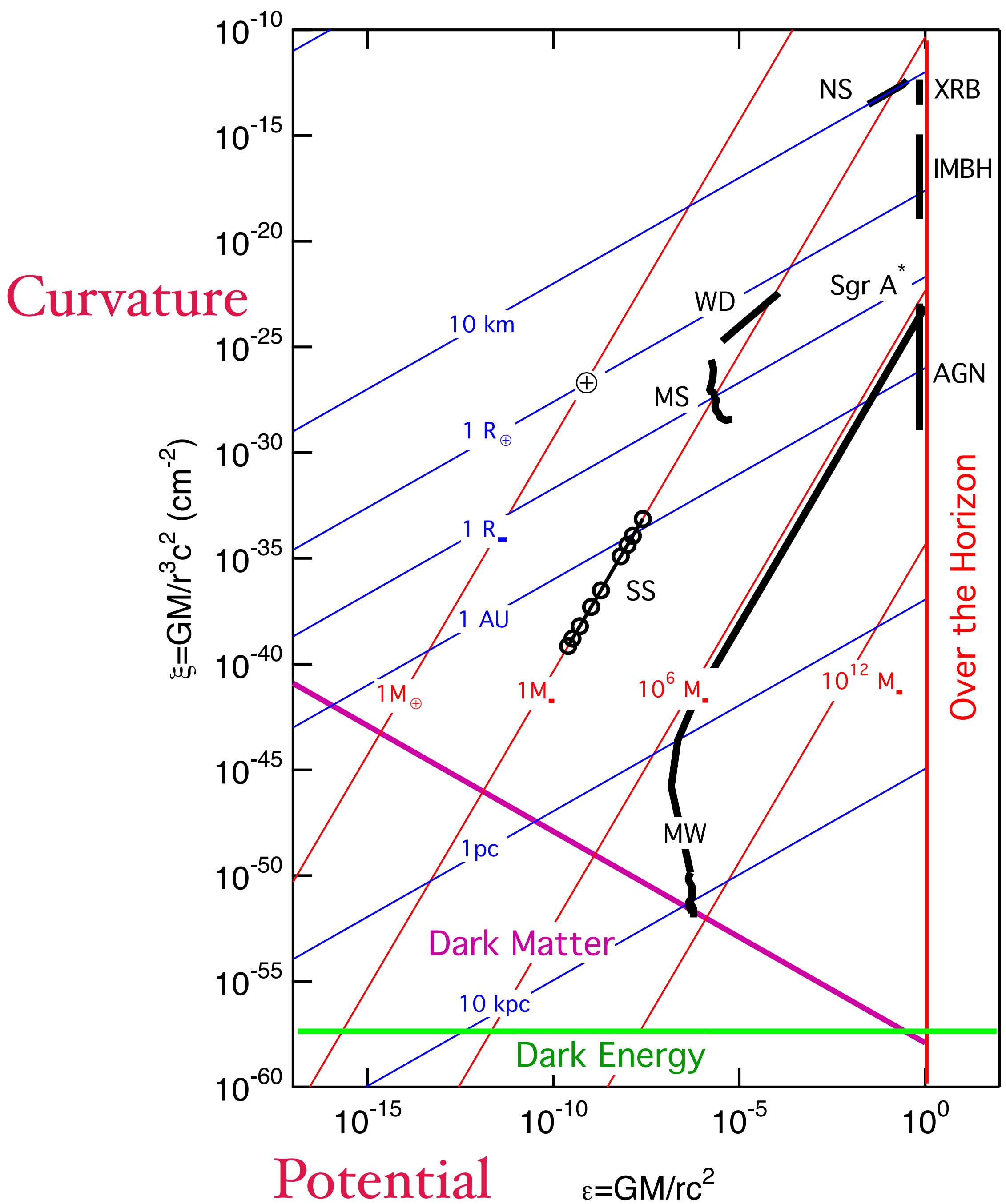


Figure 1: A parameter space for quantifying the strength of a gravitational field. The x-axis measures the potential $\epsilon \equiv GM/rc^2$ and the y-axis measures the spacetime curvature $\xi \equiv GM/r^3c^2$ of the gravitational field at a radius r away from a central object of mass M. These two parameters provide two different quantitative measures of the strength of the gravitational fields. The various curves, points, and legends are described in the text.

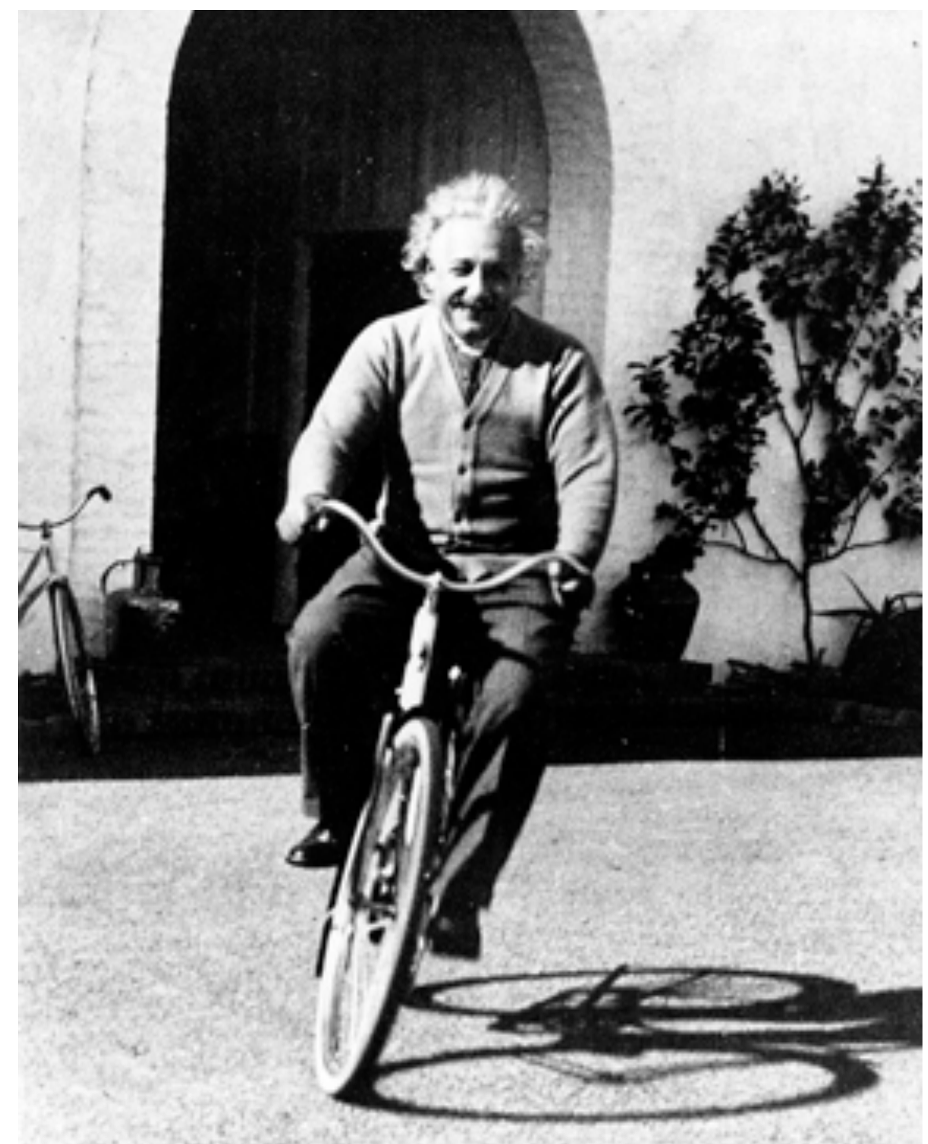
Gravity has only been tested over special ranges of scales and curvatures

e.g. Weinberg's nonlinear Quantum Mechanics- constructing to test linearity of QM

D. Psaltis, Living Reviews

UNIQUENESS OF GR

Why is General Relativity so special?



1. GR is Diffeomorphism Invariant

i.e. it exhibits 4 local symmetries - General Coordinate Transformations

$$x^\mu \to x^\mu(x')$$

Every theory can be written in a coordinate invariant way, but there is usually a preferred system of coordinates/frame of reference

- in GR there is no preferred system in the absence of matter
  - in the presence of matter there is a preferred reference frame, e.g. the rest frame of the cosmic microwave background

2. In GR Gravity is described by the curvature of spacetime

Einsteins equations take the form:

Curvature of spacetime $\propto$ Energy Momentum Density

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

radius of curvature^2 $\propto$ 1/energy density

3. GR is locally Lorentz Invariant

Every geometry is locally Minkowski -

GR can be rewritten as spin-two perturbations around Minkowski

E.o.Ms for GR are Lorentz invariant to all orders

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

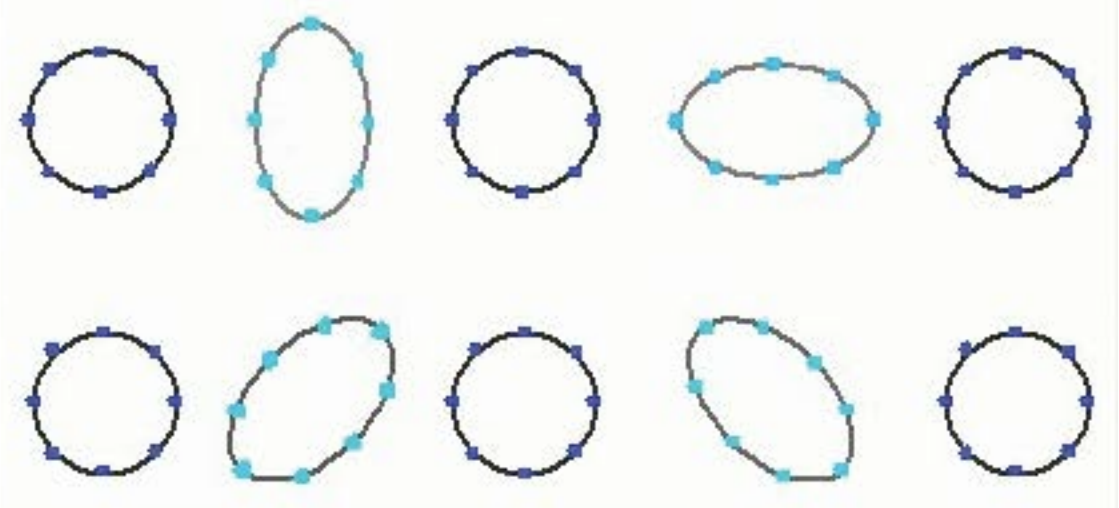
$$h_{\mu\nu} \sim R_{\mu\nu\alpha\beta}(x_P)(x^\alpha - x_P^\alpha)(x^\beta - x_P^\beta) + \mathcal{O}((x - x_P)^3)$$

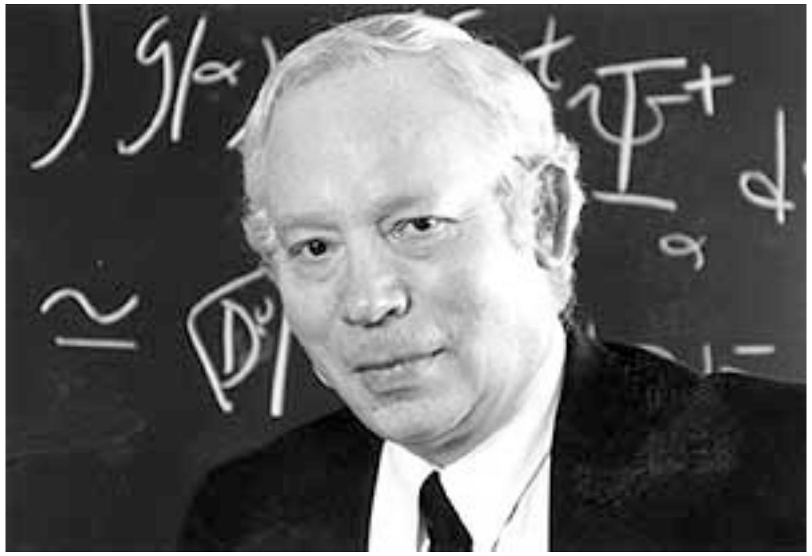
Essentially a different phrasing of the equivalence principle - ability to choice locally inertial frames

4. GR is unique theory of a massless spin-two field

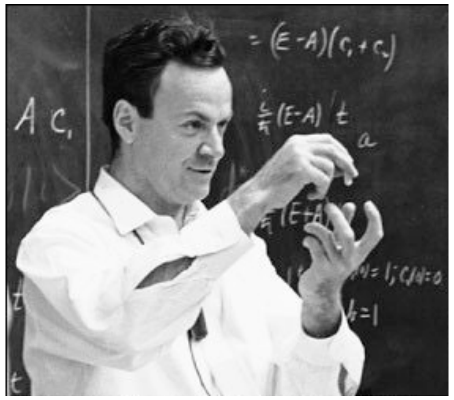
Metric perturbations transform as massless fields of spin 2!!

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$





There are only two physical polarizations of gravitational waves!



Sketch of proof

Spin 2 field is encoded in a 10 component symmetric tensor

$$h_{\mu\nu}$$

But physical degrees of freedom of a massless spin 2 field are d.o.f. = 2

We need to subtract 8 = 2 x 4

This is achieved by introducing 4 local symmetries

Every symmetry removes one component since 1 is pure gauge and the other is fixed by associated first class constraint (Lagrangian counting)

Sketch of proof

Lorentz invariance demands that the 4 symmetries form a vector (there are only 2 possible distinct scalar symmetries) and so we are led to the **unique** possibility

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

We can call this linear Diff symmetry but its really just 4 U(1) symmetries, its sometimes called **spin 2 gauge invariance**

Quadratic action

Demanding that the action is **local** and starts at lowest order in derivatives (two), we are led to a unique quadratic action which respects linear diffs

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$S = \int d^4x \frac{M_P^2}{8} h^{\mu\nu} \Box (h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) + \dots$$

Where ... are terms which vanish in de Donder/harmonic gauge. It has an elegant representation with the Levi-Civita symbols .....

$$S \propto \int d^4x \epsilon^{ABCD} \epsilon^{abcd} \eta_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Spin-2 Gauge invariance of kinetic term

$$S = \int d^4x \; \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, h_{bB} \, \partial_c \partial_C h_{dD}$$

$$\delta S \sim 2 \int d^4x \; \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, h_{bB} \, \partial_c \partial_C \, \delta h_{dD}$$

$$\delta h_{dD} = \partial_d \chi_D + d \leftrightarrow D$$

$$\delta S \sim 4 \int d^4x \; \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, h_{bB} \, \partial_c \partial_C \partial_d \chi_D = 0$$

Unique result

Most complete proof Wald 1986

There are only two nonlinear extensions of the linear Diff symmetry, (assumption over number of derivatives)

1. Linear Diff -> Linear Diff

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

2. Linear Diff -> Full Diffeomorphism

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega} \partial_\nu \xi^\omega + g_{\omega\nu} \partial_\mu \xi^\omega$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Metric emerges as derived concept

Punch Line

Massless Spin 2 = Symmetric tensor + Gauge Symmetry

↓

Nonlinear Spin 2 = Metric + Diffeomorphism Invariance

↓

Geometry!!!!

REVIEW OF MASSIVE GRAVITY/BIGRAVITY

Basic Question

What happens if we repeat this arguments starting with the assumption of a

massive spin 2 field?

i.e. suppose that the graviton is massive, are we inevitably led to the Einstein-Hilbert action (plus mass term)?

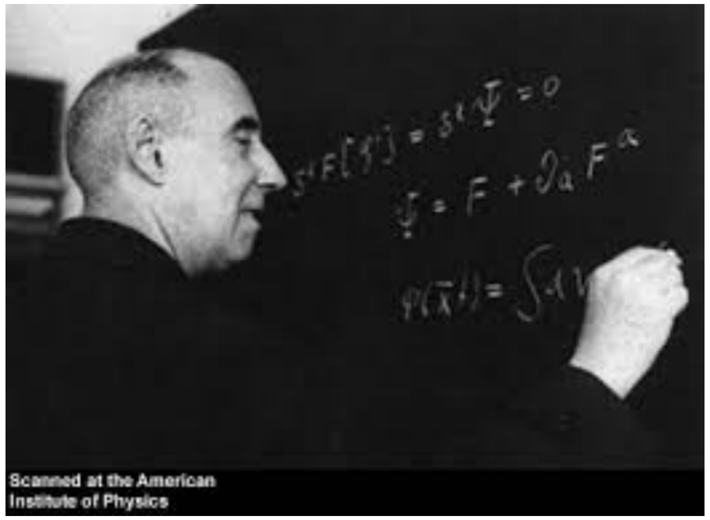
A toy example, Massive spin-1
Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2 A_\mu A^\mu$$

Unitary gauge formulation for massive spin-1 particle

Canonical Analysis shows presence of second class constraint - can analyse this way but better to reformulate as a first class constraint - reintroduce broken gauge symmetry!

Stuckelberg picture



Easiest to understand in the Stuckelberg picture in which reintroduce gauge invariance by means of a field redefinition

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 (A_\mu + \partial_\mu \chi)^2$$

Massive theory is now gauge invariant

$$A_\mu \to A_\mu + \partial_\mu \xi \,,\; \chi \to \chi - \xi$$

Therefore number of degrees of freedom are

2 A_μ + 1 χ

Free massive spin 2

In this case we should Stuckelberg the linear Diff symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

If we choose the massless kinetic term, Stuckelberg fields do not enter

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$$

There is a unique quadratic mass term

$$\mathcal{L}^{\text{unitary}}_{\text{mass}} \sim m^2 \, \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, \eta_{bB} \, h_{cC} \, h_{dD}$$

Reason: Part quadratic in X

$$\sim m^2 \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} \eta_{bB} \underbrace{\partial_c X_d}_{} \underbrace{\partial_C X_D}_{}$$

antisymmetric

$$X_\mu = \frac{1}{m} A_\mu$$

$$\sim F_{\mu\nu}^2 \; !$$

Kinetic term for helicity 1.

Part linear in X

$$\sim m^2 \, \varepsilon \, \varepsilon \, \eta \, \eta \, h \, \partial X$$

$$\sim m \, \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, \eta_{bB} \, h_{cC} \, \partial_d A_D$$

Not U(1) gauge invariant

$$A_\mu \to A_\mu + \frac{1}{m} \partial_\mu \pi$$

helicity zero mode

Final term

$$\sim \varepsilon \varepsilon \eta \eta h \partial \partial \Pi$$

$$\equiv \varepsilon \varepsilon \eta \eta \Pi \partial \partial h$$

Mixing term

Removed with de-mixing $\quad h_{\mu\nu} \to h_{\mu\nu} + \# \Pi \eta_{\mu\nu}$

Summary ≡ Fierz-Pauli

Unitary gauge

$$\mathcal{L} = \varepsilon\,\varepsilon\,\eta\, h\,\partial\partial h + m^2 \varepsilon\,\varepsilon\,\eta\,\eta\, h h$$

Stuckelberg

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{\partial_\mu A_\nu + \partial_\nu A_\mu}{m} + \frac{\partial_\mu \partial_\nu \pi}{m^2}$$

helicity 2 — helicity ±1 — helicity 0

What does massive gravity mean?

In SM, Electroweak symmetry
is spontaneously broken by the VEV of the Higgs field

$$SU(2) \times U(1)_{\rm Y} \to U(1)_{\rm EM}$$

Result, W and Z bosons become massive

Would-be-Goldstone-mode in Higgs field becomes
Stuckelberg field which gives boson mass

Higgs Vev   Higgs Boson   Stuckelberg field

$$\Phi = (v + \rho) e^{i\pi}$$

e.g. for Abelian Higgs

$$A_\mu \to A_\mu + \partial_\mu \chi \qquad \rho \to \rho + \chi$$

Symmetry Breaking Pattern

In **Massive Gravity** - Local Diffeomorphism Group and an additional global Poincare group is broken down the diagonal subgroup

$$Diff(M) \times \text{Poincare} \rightarrow \text{Poincare}_{\text{diagonal}}$$

In **Bigravity** - Two copies of local Diffeomorphism Group are broken down to a single copy of Diff group

$$Diff(M) \times Diff(M) \rightarrow Diff(M)_{\text{diagonal}}$$

Higgs for Gravity

Despite much *blood, sweat and tears* an explicit Higgs mechanism for gravity is not known

However if such a mechanism exists, we DO know how to write down the low energy effective theory in the spontaneously broken phase

For Abelian Higgs this corresponds to integrating out the Higgs boson and working at energy scales lower that the mass of the Higgs boson

Higgs Boson → ρ; Stuckelberg field → π

$$E \ll m_\rho \qquad \Phi = (v+\rho)e^{i\pi}$$

⇒ Stuckelberg formulation of massive vector bosons

Stuckelberg Formulation for Massive Gravity

Arkani-Hamed et al 2002
de Rham, Gabadadze 2009

Diffeomorphism invariance is spontaneously broken but maintained by introducing Stueckelberg fields

Vev of spin 2 Higgs field defines a 'reference metric'

$$f_{\mu\nu} = \langle \hat{O}_{\mu\nu} \rangle$$

reference metric

Stuckelberg fields

Dynamical Metric

$$g_{\mu\nu}(x)$$

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B$$

helicity-1 mode of graviton

$$\phi^a = x^a + \frac{1}{mM_P} A^a + \frac{1}{\Lambda^3}\partial^a \pi$$

$$\Lambda^3 = m^2 M_P$$

helicity-0 mode of graviton

Helicity Zero mode = Galileon

The helicity zero mode $\pi(x)$ only enters in the combination

$$\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi(x)$$

This is invariant under the
global nonlinearly realized symmetry

$$\pi(x) \to \pi(x) + c + v_\mu x^\mu$$

$$\Pi_{\mu\nu} \to \Pi_{\mu\nu}$$

Galilean Operators

$$\mathcal{L}_1 = \Pi \;\; \varepsilon \;\; \varepsilon \;\; \eta \;\; \eta \;\; \eta \;\; \partial\partial\Pi$$

$$\mathcal{L}_3 = \Pi \;\; \varepsilon \;\; \varepsilon \;\; \eta \;\; \eta \;\; \partial\partial\Pi \;\; \partial\partial\Pi$$

$$\mathcal{L}_4 = \Pi \;\; \varepsilon \;\; \varepsilon \;\; \eta \;\; \partial\partial\Pi \;\; \partial\partial\Pi \;\; \partial\partial\Pi$$

$$\mathcal{L}_5 = \Pi \;\; \varepsilon \;\; \varepsilon \;\; \partial\partial\Pi \;\; \partial\partial\Pi \;\; \partial\partial\Pi \;\; \partial\partial\Pi$$

Galileon - helicity 2 interactions

$$\mathcal{L}_2' = \varepsilon \; \varepsilon \; \eta \; \eta \; \partial\partial\pi \; h$$

$$\mathcal{L}_3' = \varepsilon \; \varepsilon \; \eta \; \partial\partial\pi \; \partial\partial\pi \; h$$

$$\mathcal{L}_4' = \varepsilon \; \varepsilon \; \partial\partial\pi \; \partial\partial\pi \; \partial\partial\pi \; h$$

'Characteristic polynomials'

$$\det\left(\alpha h + \beta\, \partial\partial\pi + \gamma\, \eta\right)$$

Discovering how to square root

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B$$

$$\phi^a = x^a + \frac{1}{mM_P}A^a + \frac{1}{\Lambda^3}\partial^a\pi$$

Helicity zero mode enters reference metric **squared**

$$F_{\mu\nu} \approx \eta_{\mu\nu} + \frac{2}{\Lambda^3}\partial_\mu\partial_\nu\pi + \frac{1}{\Lambda^6}\partial_\mu\partial_\alpha\pi\partial^\alpha\partial_\nu\pi$$

To extract dominant helicity zero interactions we need to take a square root

$$\left[\sqrt{g^{-1}F}\right]_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{\Lambda^3}\partial_\mu\partial_\nu\pi$$

Branch uniquely chosen to give rise to 1 when Minkowski de Rham, Gabadadze, AJT 2010

Hard Λ_3 Massive Gravity

$$Diff(M) \times \text{Poincare} \to \text{Poincare}_{\text{diagonal}}$$

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(M_P^2 R[g] - m^2 \sum_{n=0}^{4} \beta_n \mathcal{U}_n\right) + \mathcal{L}_M$$

$$K = 1 - \sqrt{g^{-1} f}$$

$$\text{Det}[1 + \lambda K] = \sum_{n=0}^{d} \lambda^n \mathcal{U}_n(K)$$

Characteristic Polynomials

Double epsilon structure!!!!!

Unique low energy EFT where the strong coupling scale is

$$\Lambda_3 = (m^2 M_P)^{1/3}$$

5 propagating degrees of freedom

5 polarizations of gravitational waves!!!!

Hassan, Rosen 2011

Hard Massless plus Λ_3 Massive Gravity

$$Diff(M) \times Diff(M) \to Diff(M)_{\rm diagonal}$$

$$\mathcal{L} = \frac{1}{2}\left(M_P^2 \sqrt{-g} R[g] + M_f^2 \sqrt{-f} R[f] - m^2 \sum_{n=0}^{d} \beta_n U_n(K) \right) + \mathcal{L}_M$$

$$\mathrm{Det}[1+\lambda K] = \sum_{n=0}^{d} \lambda^n \mathcal{U}_n(K)$$

$$K = 1 - \sqrt{g^{-1} f}$$

decoupling limit $\downarrow$ $M_f \to \infty$

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(M_P^2 R[g] - m^2 \sum_{n=0}^{4} \beta_n \mathcal{U}_n \right) + \mathcal{L}_M$$

+decoupled massless graviton $f_{\mu\nu}$

Bigravity=
massless graviton (2 d.o.f.)
+ massive graviton (5 d.o.f.)

EMERGENCE OF GALILEONS

Universal Decoupling Limit: Galileon

At energies $m \ll E \ll M_{\rm PLanck}$ $\Lambda_3 = (m^2 M_{\rm Planck})^{1/3}$

All Lorentz invariant Hard and Soft and Multi-graviton theories look like **Galileon theories** (plus massless spin 2 plus Maxwell)

$$\pi \to \pi + v_\mu x^\mu + c \qquad K_{\mu\nu} = \frac{\partial_\mu \partial_\nu \pi}{\Lambda_3^3}$$

$$K = 1 - \sqrt{g^{-1} f}$$

$$\Lambda^4 L_0 = \left[\frac{M^2}{2} R - \Lambda^3 M \sum_n \alpha_n \mathcal{E}\mathcal{E} g^{4-n} K^n\right] + \Lambda^4 \sum \beta_{p,q,r} \left(\frac{\nabla}{\Lambda}\right)^p K^q_{\mu\nu} \left(\frac{R_{\mu\nu\rho\sigma}}{\Lambda^2}\right)^r$$

Stuckelberg Formulation for Bigravity

Fasiello, AJT 1308.1647

$$\mathcal{L} = \frac{1}{2}\left(M_P^2 \sqrt{-g} R[g] + M_f^2 \sqrt{-f} R[f] - m^2 \sum_{n=0}^{d} \beta_n U_n(K) \right) + \mathcal{L}_M$$

Dynamical metric I

$$g_{\mu\nu}(x)$$

Dynamical metric II

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B$$

$$\phi^A = x^A + \frac{1}{\Lambda_3^3}\partial^A \pi$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu} \quad f_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} v_{\mu\nu}$$

Explicitly Decoupling limit for Bigravity

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu} \qquad f_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} v_{\mu\nu}$$

de Rham, Gabadadze 2009

Fasiello, AJT 2013

massless helicity 2

massless helicity 0

$$S_{\text{helicity}-2/0} = \int \mathrm{d}^4x \Big[-\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{1}{4} v^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} v_{\alpha\beta} + \frac{\Lambda_3^3}{2} h^{\mu\nu}(x) X^{\mu\nu} + \frac{M_{\rm p}\Lambda_3^3}{2M_{\rm f}} v_{\mu A}[x^a + \Lambda_3^{-3}\partial^a \pi](\eta_\nu^A + \Pi_\nu^A) Y^{\mu\nu} \Big]$$

$$X^{\mu\nu} = -\frac{1}{2} \sum_{n=0}^{4} \frac{\hat{\beta}_n}{(3-n)!n!} \varepsilon^{\mu\dots}\varepsilon^{\nu\dots}(\eta + \Pi)^n \eta^{3-n}$$

$$\Pi_{ab} = \frac{\partial_a \partial_b \pi}{\Lambda_3^3}$$

$$Y^{\mu\nu} = -\frac{1}{2} \sum_{n=0}^{4} \frac{\hat{\beta}_n}{(4-n)!(n-1)!} \varepsilon^{\mu\dots}\varepsilon^{\nu\dots}(\eta + \Pi)^{(n-1)} \eta^{4-n}$$

Post-diagonalization: Galileons

de Rham, Gabadadze 2009
Fasiello, AJT 2013

$$S = \int d^4x \left[-\frac{1}{4} h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{1}{4} v^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} v_{\alpha\beta} \right] + S_{\text{Galileon}} + S_{\text{mattercoupling}}$$

$$S_{\text{Galileon}} = \sum_{n=0}^{4} \pi \, c_n \, \mathcal{U}_n(K) \qquad \text{Det}[1 + \lambda K] = \sum_{n=0}^{d} \lambda^n \mathcal{U}_n(K)$$

$$K^\mu_\nu = \partial^\mu \partial_\nu \pi$$

Novel feature, matter has `disformal' couplings

$$S_{\text{matter coupling}} = \int d^4x \frac{1}{M_P} (\pi T + \partial_\mu \pi \partial_\nu \pi T^{\mu\nu} + \dots)$$

GALILEON DUALITY (OLD STORY)

 = 

Fasiello, AJT 1308.1647

There are two ways to introduce Stuckelberg fields!

Dynamical metric I

$$g_{\mu\nu}(x)$$

Dynamical metric II

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B$$

$$\tilde{x}^A = \phi^A(x) = x^A + \partial^A\pi(x)$$

OR

Dynamical metric I

$$\tilde{G}_{AB}(\tilde{x}) = g_{\mu\nu}(Z)\partial_A Z^\mu \partial_B Z^\nu$$

Dynamical metric II

$$f_{AB}(\tilde{x})$$

$$x^\mu = Z^\mu(\tilde{x}) = \tilde{x}^\mu + \partial^\mu\tilde{\pi}(\tilde{x})$$

Galileon Duality!!!!!

Dual Galileons fields

\`Galileon Duality' - de Rham, Matteo Fasiello, AJT 2013
Curtright and Fairlie arXiv:1212.6972

For every **Galileon** field $\pi(x)$
define the **Dual Galileon** field via the implicit field redefinition

$$\tilde{x}^A = \phi^A(x) = x^A + \partial^A \pi(x)$$

$$x^\mu = Z^\mu(\tilde{x}) = \tilde{x}^\mu + \partial^\mu \rho(\tilde{x})$$

$$\pi(x) = -\rho(\tilde{x}) - \frac{1}{\Lambda^3}(\tilde{\partial}\rho(\tilde{x}))^2 \qquad \rho(\tilde{x}) = -\pi(x) + \frac{1}{\Lambda^3}(\partial\pi(x))^2$$

Dual Galileons fields

$$\tilde{x}^A = \phi^A(x) = x^A + \partial^A \pi(x)$$

$$x^\mu = Z^\mu(\tilde{x}) = \tilde{x}^\mu + \partial^\mu \rho(\tilde{x})$$

$$\pi(x) = -\rho(\tilde{x}) - \frac{1}{\Lambda^3}(\tilde{\partial}\rho(\tilde{x}))^2 \qquad \rho(\tilde{x}) = -\pi(x) + \frac{1}{\Lambda^3}(\partial\pi(x))^2$$

Explicitly this is

$$\rho(x) = -\pi(x) + \frac{1}{2}(\partial\pi)^2 - \frac{1}{2}\partial^a\pi\partial^b\pi\partial_a\partial_b\pi + \text{infinite number of terms}\ldots$$

or for spherical symmetry

$$\rho(r) = -\pi(r) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \partial_r^{n-2}\left((\partial_r \pi)^n\right)$$

Dual Galileons Lagrangians

Galileon operators: $\mathcal{L}_n(\pi) = \pi\epsilon\epsilon(\partial\partial\pi)^{n-1}\eta^{D-n+1}$

For every **Galileon** field Lagrangian in D spacetime dimensions

$$\mathcal{L}(\pi) = c_2\mathcal{L}_2(\pi) + c_3\mathcal{L}_3(\pi) + c_4\mathcal{L}_4(\pi) + \dots$$

admits a dual formulation as a Galileon

$$\mathcal{L}(\rho) = p_2\mathcal{L}_2(\rho) + p_3\mathcal{L}_3(\rho) + p_4\mathcal{L}_4(\rho) + \dots$$

with distinct coefficients

$$p_n = \frac{1}{n}\sum_{k=2}^{D+1}(-1)^k c_k \frac{k(d-k+1)!}{(n-k)!(d-n+1)!}$$

Galileon Symmetry is Translation Symmetry (for the other metrics coordinates)

Diff(M) Stuckelberg

$$\phi^A = x^A + \partial^A \pi$$

Galileon

$$+$$

Galileon transformation

$$\pi(x) \to \pi(x) + c + v_\mu x^\mu$$

$$\Downarrow$$

$$\phi^A \to \phi^A + v^A$$

Translation in ϕ^A *!*

Dual Formulation

$$\rho(\tilde{x}) \to \rho(\tilde{x}) + u_\mu \tilde{x}^\mu \quad \Rightarrow \quad x^\mu \to x^\mu + u^\mu$$

Dual of a Free theory

A **free theory** in Minkowski spacetime which is a causal theory with an analytic S-matrix with no strong coupling issues and is UV-complete is dual to a **quintic** Galileon theory.

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 \equiv -\frac{1}{2}(\partial\rho)^2 - \frac{1}{6}\mathcal{L}_3(\rho) - \frac{1}{8}\mathcal{L}_4(\rho) - \frac{1}{30}\mathcal{L}_5(\rho)$$

Field redefinitions do NOT change physics - even at quantum level

Duality works with Matter!

Any local coupling to matter maps to a local coupling to matter in the dual theory

Example

$$S_{\rm matter} = \int d^4x \left(g\pi(x)\chi(x) - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 \right)$$

is dual to

$$S_{\rm matter} = \int d^4x \sqrt{-G} \left(-g \left(\rho(x) + \frac{1}{\Lambda^3}(\partial\rho)^2 \right) \zeta(x) - \frac{1}{2} G^{\mu\nu}(\partial_\mu\zeta\partial_\nu\zeta) - \frac{1}{2}m^2\zeta^2 \right)$$

where

$$\zeta(\tilde{x}) = \chi(x) \quad \Sigma_{\mu\nu} = \frac{1}{\Lambda^3}\partial_\mu\partial_\nu\rho$$

$$G_{\mu\nu} = \eta^{\alpha\beta}(\eta_{\mu\alpha} + \Sigma_{\mu\alpha})(\eta_{\nu\beta} + \Sigma_{\nu\beta})$$

GALILEON DUALITY (NEW STORY)

Differential Geometry shorthand

Introduce shorthand

$$\varepsilon ABC.... = \varepsilon_{a_1 a_2 a_2 ...} A^{a_1} \wedge B^{a_2} \wedge C^{a_3} \ldots .$$

For Lorentz index valued one-forms

$$A_a = A_{\mu a} \mathrm{d}x^\mu$$

So for example:

$$\eta_{\mu a} \mathrm{d}x^\mu = \mathrm{d}x_a \qquad \partial_\mu \partial_a \pi \mathrm{d}x^\mu = \mathrm{d}\partial_a \pi .$$

Galileon Action

In this shorthand notation, the Galileon action in d spacetime dimensions is

$$S_1 = \int \left[\sum_{n=0}^{d} c_n \pi \, \varepsilon (\mathrm{d}x)^{d-n} (\mathrm{d}\partial\pi)^n \right]$$

Principle advantage: Although defining a theory on Minkowski, notation is independent of coordinate system and hence accommodates field dependent coordinate transformations

Galileon Duality (again)

$$\tilde{\pi}(\tilde{x}) = \pi(x) + \frac{1}{2}\lambda\partial_\mu\pi(x)\partial^\mu\pi(x)$$
$$\tilde{x}^\mu = x^\mu + \lambda\partial^\mu\pi(x)\,.$$

$$\tilde{\partial}_\mu\tilde{\pi} = \partial_\mu\pi$$

$$X = -\tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi$$

$$S_1 = \int\left[\sum_{n=0}^{d} c_n\pi\,\varepsilon(\mathrm{d}x)^{d-n}(\mathrm{d}\partial\pi)^n\right]$$

$$\updownarrow$$

$$S_1 = \int\left[\sum_{n=0}^{d} c_n(\tilde{\pi}+\lambda\tilde{X})\,\varepsilon(\mathrm{d}\tilde{x}-\lambda\mathrm{d}\tilde{\partial}\tilde{\pi})^{d-n}(\mathrm{d}\tilde{\partial}\tilde{\pi})^n\right]$$

Generalisation to Generic Scalar Theories (in Minkowski)

$$X = -\frac{1}{2}(\partial\phi)^2$$

Consider the general scalar theories leading to second order equations of motion (Horndeski with frozen metric)

$$S = \int \left[\sum_{n=0}^{d} \sum_{p,q=0}^{1} F_{npq}(\phi, X)\varepsilon(\mathrm{d}x)^{d-n-p-q}(\mathrm{d}\partial\phi)^n(\mathrm{d}\phi\partial\phi)^p(\mathrm{d}X\partial\phi)^q \right]$$

Consider a field dependent diffeomorphism

$$\tilde{x}^\mu = x^\mu + G(\phi, X)\partial^\mu\phi$$

$$\mathrm{d}\tilde{x} = \mathrm{d}x + G(\mathrm{d}\partial\phi) + G_{,\phi}(\mathrm{d}\phi\partial\phi) + G_{,X}(\mathrm{d}X\partial\phi)$$

$$X = -\frac{1}{2}(\partial\phi)^2$$

Field redefinitions

$$S = \int \left[\sum_{n=0}^{d} \sum_{p,q=0}^{1} F_{npq}(\phi, X)\varepsilon(\mathrm{d}x)^{d-n-p-q}(\mathrm{d}\partial\phi)^n(\mathrm{d}\phi\partial\phi)^p(\mathrm{d}X\partial\phi)^q \right]$$

$$\tilde{\phi} = \phi + F(\phi, X) \qquad \tilde{X} = -\frac{1}{2}(\tilde{\partial}\tilde{\phi})^2 = W(\phi, X)^2 X$$

$$\tilde{\partial}_\mu\tilde{\phi} = W(\phi, X)\partial_\mu\phi \qquad \tilde{x}^\mu = x^\mu + G(\phi, X)\partial^\mu\phi$$

$$\mathrm{d}\tilde{x} = \mathrm{d}x + G(\mathrm{d}\partial\phi) + G_{,\phi}(\mathrm{d}\phi\partial\phi) + G_{,X}(\mathrm{d}X\partial\phi)$$

$$\mathrm{d}\tilde{X} = W^2\mathrm{d}X + 2WW_{,X}X\mathrm{d}X + 2WW_{,\phi}X\mathrm{d}\phi \qquad \mathrm{d}\tilde{\phi} = \mathrm{d}\phi + F_{,\phi}\mathrm{d}\phi + F_{,X}\mathrm{d}X$$

$$\mathrm{d}\tilde{\phi}\tilde{\partial}\tilde{\phi} = W(\phi, X)\left(\mathrm{d}\phi\partial\phi + F_{,\phi}\mathrm{d}\phi\partial\phi + F_{,X}\mathrm{d}X\partial\phi\right)$$

Inverse Duality Transformation = Duality Transformation

$$X = -\frac{1}{2}(\partial\phi)^2$$

$$\phi = \tilde{\phi} - F(\phi, X) = \tilde{\phi} + \tilde{F}(\tilde{\phi}, \tilde{X})\,,$$

$$\partial_\mu \phi = \frac{1}{W(\phi, X)}\tilde{\partial}_\mu \tilde{\phi} = \tilde{W}(\tilde{\phi}, \tilde{X})\tilde{\partial}_\mu \tilde{\phi}\,,$$

$$x^\mu = \tilde{x}^\mu - G(\phi, X)\partial^\mu \phi = \tilde{x}^\mu - \frac{G(\phi, X)}{W(\phi, X)}\tilde{\partial}^\mu \tilde{\phi} = \tilde{x}^\mu + \tilde{G}(\tilde{\phi}, \tilde{X})\tilde{\partial}^\mu \tilde{\phi}$$

Integrability Condition

Consistency of the transformations

$$X = -\frac{1}{2}(\partial\phi)^2$$

$$\mathrm{d}\tilde{\phi} = \tilde{\partial}_\mu \tilde{\phi}(\tilde{x})\mathrm{d}\tilde{x}^\mu$$

Imposes an integrability condition of the naively 3 free functions W,G,F

$$W\mathrm{d}\phi - 2WX\mathrm{d}G - WG\mathrm{d}X = \mathrm{d}\phi + \mathrm{d}F$$

Or equivalently

$$\mathrm{d}(F + 2WXG) = W\mathrm{d}\phi + WG\mathrm{d}X + 2\mathrm{d}WXG$$

This leads to a system of two dimensional differential equations - they admit a solution only if the following integrability condition is satisfied

$$\mathrm{d}(W\mathrm{d}\phi + WG\mathrm{d}X + 2\mathrm{d}WXG) = 0$$

$$\partial_X W + 2\partial_\phi W(G + X\partial_X G) = \partial_\phi(WG) + 2\partial_X WX\partial_\phi G$$

Recipe

Duality Transformation

$$\tilde{\phi} = \phi + F(\phi, X)$$

$$\tilde{X} = -\frac{1}{2}(\tilde{\partial}\tilde{\phi})^2 = W(\phi, X)^2 X$$

$$\tilde{\partial}_\mu \tilde{\phi} = W(\phi, X)\partial_\mu \phi$$

$$\tilde{x}^\mu = x^\mu + G(\phi, X)\partial^\mu \phi$$

1. Specify G
2. Solve integrability condition for W

$$\partial_X W + 2\partial_\phi W(G + X\partial_X G) = \partial_\phi(WG) + 2\partial_X W X \partial_\phi G$$

3. Solve equation for F

$$\mathrm{d}(F + 2WXG) = W\mathrm{d}\phi + WG\mathrm{d}X + 2\mathrm{d}WXG$$

$$S = \int \left[\sum_{n=0}^{d} \sum_{p,q=0}^{1} F_{npq}(\phi, X)\varepsilon(\mathrm{d}x)^{d-n-p-q}(\mathrm{d}\partial\phi)^n(\mathrm{d}\phi\partial\phi)^p(\mathrm{d}X\partial\phi)^q \right]$$

Example: DBI-Galileon Duality

If $W = 1$ then the integrability condition reduces to

$$\partial_\phi G = 0 \qquad \text{i.e. } G = G(X)$$

And the equation for F is solved by

$$F = F(X) = \int_0^X \mathrm{d}G(X')\mathrm{d}X - 2XG$$

As a special example consider

$$G(X) = \lambda\gamma = \lambda \frac{1}{\sqrt{1-2X}} \qquad F(X) = \lambda\left(1 - \frac{1}{\sqrt{1-2X}}\right) = \lambda(1-\gamma)$$

$$\gamma = \frac{1}{\sqrt{1-2X}} = \frac{1}{\sqrt{1+(\partial\phi)^2}}$$

Example: DBI-Galileon Duality

Tasinato 2016

$$\tilde{\phi} = \phi + \lambda(1-\gamma) \qquad \tilde{\partial}\tilde{\phi} = \partial\phi$$

$$\tilde{x}^{\mu} = x^{\mu} + \lambda\gamma\partial^{\mu}\phi \, , \qquad \tilde{\gamma} = \gamma ;$$

$$S = \int \gamma^{-1} \sum_{n=0}^{d} C_n \varepsilon (\mathrm{d}x)^{d-n} (\mathrm{d}(\gamma\partial\phi))^n$$

$$S = \int \tilde{\gamma}^{-1} \sum_{n=0}^{d} C_n \varepsilon (\mathrm{d}\tilde{x} - \lambda\tilde{\gamma}\mathrm{d}\tilde{\partial}\tilde{\phi})^{d-n} (\mathrm{d}(\tilde{\gamma}\tilde{\partial}\tilde{\phi}))^n \, ,$$

$$= \int \tilde{\gamma}^{-1} \sum_{n=0}^{d} \tilde{C}_n(\lambda) \varepsilon (\mathrm{d}\tilde{x})^{d-n} (\mathrm{d}(\tilde{\gamma}\tilde{\partial}\tilde{\phi}))^n$$

$$\tilde{C}_n(\lambda) = \sum_{r=0}^{n} (-\lambda)^r \frac{(d-n+r)!}{r!(d-n)!} C_{n-r} \, .$$

Coupling to Massless Gravity?

- So far we have only defined the duality for scalar fields in Minkowski spacetime

- Since the duality is a field dependent diffeomorphism, a naive covariantization (covariant Galileon/Horndeski) will violate duality

- Solution appears to be to work in first order Einstein-Cartan formulation where spin-connection is independent

Coupling to Massless Gravity?

To see why, consider Minkowski spacetime in vierbein notation

$$e^a = \mathrm{d}x^a \qquad\qquad g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \mathrm{d}x^\mu \mathrm{d}x^\mu$$

Under the (Minkowski) duality transformation

$$\tilde{e}^a = \mathrm{d}\tilde{x}^a = \mathrm{d}x^a + \mathrm{d}(G(\phi, X)\partial^a \phi) = e^a + \mathrm{d}(G(\phi, X)\partial^a \phi)$$

This looks like a local field redefinition of the vierbein!!!

Generalisation with dynamical gravity

Define a Lorentz vector/diff scalar $\Phi^a = e^\mu_a \partial_\mu \phi$

such that $$e^a \Phi_a = \mathrm{d}\phi$$

Define a Lorentz scalar/diff scalar $$X = -\frac{1}{2}\eta_{ab}\Phi^a\Phi^b$$

Define a spin-connection via a covariant derivative

$$D[\omega]\Phi^a = \mathrm{d}\Phi^a + \omega^a{}_b\Phi^b$$

Generalisation with dynamical gravity

$$X = -\frac{1}{2}\eta_{ab}\Phi^a\Phi^b$$

Then the duality transformation is a local field redefinition on the scalar and vierbein of the form

$$\tilde{\phi} = \phi + F(\phi, X)$$

$$\tilde{e}^a = e^a + D[\omega]\left(G(\phi, X)\Phi^a\right)$$

$$\tilde{\Phi}^a = W(\phi, X)\Phi^a \Rightarrow \tilde{X} = W(\phi, X)^2 X .$$

Condition for invertibility

Requiring that the inverse of the duality transformation is also a duality transformation

$$
\begin{aligned}
e^a &= \tilde{e}^a - D[\omega]\left(G(\phi, X)\Phi^a\right) = \tilde{e}^a - D[\omega]\left(G(\phi, X)W(\phi, X)^{-1}\tilde{\Phi}^a\right) \\
&= \tilde{e}^a + D[\tilde{\omega}]\left(\tilde{G}(\tilde{\phi}, \tilde{X})\tilde{\Phi}^a\right) .
\end{aligned}
$$

Imposes that

$$
\tilde{\omega}^{ab} = \omega^{ab}\,, \quad \tilde{G}(\tilde{\phi}, \tilde{X}) = -G(\phi, X)W(\phi, X)^{-1}
$$

Hence the spin-connection cannot remain the usual torsionless one (first order formulation necessary!!) since it does not transform when the vierbein does

Integrability Condition

Requiring that $$\tilde{e}^a \tilde{\Phi}_a = \mathrm{d}\tilde{\phi}$$

Leads to the same integrability condition we obtained in Minkowski spacetime

$$W \mathrm{d}\phi - 2WX\mathrm{d}G - WG\mathrm{d}X = \mathrm{d}\phi + \mathrm{d}F$$

Conclusion: Whenever there exists a duality transformation for a scalar theory in Minkowski spacetime, we can couple it to massless gravity and preserve the same symmetry.

Full Duality Transformations

$$\tilde{\phi} = \phi + F(\phi, X) \quad \tilde{e}^a = e^a + D[\omega]\left(G(\phi, X)\Phi^a\right) , \quad \tilde{\Phi}^a = W(\phi, X)\Phi^a \quad \tilde{\omega}^{ab} = \omega^{ab} ,$$

$$\phi = \tilde{\phi} + \tilde{F}(\tilde{\phi}, \tilde{X}) \quad e^a = \tilde{e}^a + D[\tilde{\omega}]\left(\tilde{G}(\tilde{\phi}, \tilde{X})\tilde{\Phi}^a\right) , \quad \Phi^a = \tilde{W}(\tilde{\phi}, \tilde{X})\tilde{\Phi}^a \quad \omega^{ab} = \tilde{\omega}^{ab}$$

$$F(\tilde{\phi}, \tilde{X}) = -F(\phi, X) , \; \tilde{G}(\tilde{\phi}, \tilde{X}) = -G(\phi, X)W(\phi, X)^{-1} , \; \tilde{W}(\tilde{\phi}, \tilde{X}) = W(\phi, X)^{-1} ,$$

Together with integrability condition

$$W\mathrm{d}\phi - 2WX\mathrm{d}G - WG\mathrm{d}X = \mathrm{d}\phi + \mathrm{d}F$$

Covariant Action

In first order form (spin-connection is independent)

$$S[e,\omega,\phi] = \int \sum_{p,q=0}^{1} \sum_{m=0}^{d-2-p-q} M_{mpq}(\phi, X)\varepsilon e^{d-2-m-p-q}(D[\omega]\Phi)^m(\mathrm{d}\phi\Phi)^p(\mathrm{d}X\Phi)^q R[\omega]$$
$$+ \int \left[\sum_{p,q=0}^{1} \sum_{n=0}^{d-p-q} F_{npq}(\phi, X)\varepsilon e^{d-n-p-q}(D[\omega]\Phi)^n(\mathrm{d}\phi\Phi)^p(\mathrm{d}X\Phi)^q\right] . \qquad ($$

Duality transformations imply

$$e = \tilde{e} + \tilde{G}(D[\tilde{\omega}]\tilde{\Phi}) + \frac{\partial \tilde{G}}{\partial \tilde{X}}(\mathrm{d}\tilde{X}\tilde{\Phi}) + \frac{\partial \tilde{G}}{\partial \tilde{\phi}}(\tilde{\mathrm{d}}\phi\tilde{\Phi}) .$$

$$D[\omega]\Phi = \tilde{W} D[\tilde{\omega}]\tilde{\Phi} + \mathrm{d}W\tilde{\Phi} = \tilde{W}(D[\tilde{\omega}]\tilde{\Phi}) + \frac{\partial \tilde{W}}{\partial \tilde{X}}(\mathrm{d}\tilde{X}\tilde{\Phi}) + \frac{\partial \tilde{W}}{\partial \tilde{\phi}}(\tilde{\mathrm{d}}\phi\tilde{\Phi}) .$$

$$(\mathrm{d}\phi\Phi) = \tilde{W}(\mathrm{d}\tilde{\phi}\tilde{\Phi}) + \tilde{W}\frac{\partial \tilde{F}}{\partial \tilde{X}}(\mathrm{d}\tilde{X}\tilde{\Phi}) + \tilde{W}\frac{\partial \tilde{F}}{\partial \tilde{\phi}}(\tilde{\mathrm{d}}\phi\tilde{\Phi}) .$$

$$(\mathrm{d}X\Phi) = \tilde{W}^3(\mathrm{d}\tilde{X}\tilde{\Phi}) + 2\tilde{W}^2\frac{\partial \tilde{W}}{\partial \tilde{X}}(\mathrm{d}\tilde{X}\tilde{\Phi}) + 2\tilde{W}^2\frac{\partial \tilde{W}}{\partial \tilde{\phi}}(\tilde{\mathrm{d}}\phi\tilde{\Phi}) ,$$

Covariant Action

In first order form (spin-connection is independent)

$$S[e,\omega,\phi] = \int \sum_{p,q=0}^{1} \sum_{m=0}^{d-2-p-q} M_{mpq}(\phi,X)\varepsilon e^{d-2-m-p-q}(D[\omega]\Phi)^m(\mathrm{d}\phi\Phi)^p(\mathrm{d}X\Phi)^q R[\omega]$$

$$+ \int \left[\sum_{p,q=0}^{1} \sum_{n=0}^{d-p-q} F_{npq}(\phi,X)\varepsilon e^{d-n-p-q}(D[\omega]\Phi)^n(\mathrm{d}\phi\Phi)^p(\mathrm{d}X\Phi)^q \right] . \qquad ($$

Duality transformations imply

$$\tilde{M}_{mpq} = \sum_{p,q=0}^{1} \sum_{m=0}^{d-2-p-q} \alpha_{mpq;m'p'q'}(\phi,X) M_{m'p'q'} ,$$

$$\tilde{F}_{mpq} = \sum_{p,q=0}^{1} \sum_{n=0}^{d-p-q} \beta_{mpq;m'p'q'}(\phi,X) F_{m'p'q'} .$$

Explicit Example

e.g. Standard cosmological theory (e.g. k-infation, k-essence)

$$
\begin{aligned}
S &= \int \sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + p(\phi, X)\right] \\
&= \int \epsilon\left[\frac{M_{\rm Pl}^2}{4}e^2 R[\omega] + \frac{1}{4!}\epsilon e^4 p(\phi, X)\right] \\
&= \int \epsilon_{abcd}\left[\frac{M_{\rm Pl}^2}{4}e^a\wedge e^b\wedge R^{cd}[\omega] + \frac{1}{4!}\epsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d p(\phi, X)\right]
\end{aligned}
$$

Maps under the duality transformation to

$$
S = \int \epsilon\left[\frac{M_{\rm Pl}^2}{4}\left(\tilde{e} + D[\tilde{\omega}](\tilde{G}\tilde{\Phi}^a)\right)^2 R[\tilde{\omega}] + \frac{1}{4!}\epsilon\left(\tilde{e} + D[\tilde{\omega}](\tilde{G}\tilde{\Phi}^a)\right)^4 \tilde{p}(\tilde{\phi}, \tilde{X})\right]
$$

$$
\tilde{p}(\tilde{\phi}, \tilde{X}) = p\left(\tilde{\phi} + F(\tilde{\phi}, \tilde{X}), \tilde{W}(\tilde{\phi}, \tilde{X})^2\tilde{X}\right)
$$

Conclusions (Old)

- Galileons emerge as decoupling limits of massive gravity and bigravity theories
- Galileon Duality maps two naive different Galileon theories into each other by means of a non-local field redefinition
- Galileon Duality preserves S-matrix

Conclusions (New)

- Galileon Duality can be generalised to **any scalar field theory** in Minkowski spacetime provided a simple integrability condition is met - *Infinite free choice of duality transformations*
- Duality transformations map class of theories that lead to second order equations of motion into each other
- Symmetry of Galileon theory is not important
- Every theory in Minkowsi that admits a duality can be extended to a covariant theory in a simple way provided we work in the first order Einstein-Cartan formalism
- Resulting theories are in general not equivalent to the Horndeski/covariant Galileon theories.
- Covariant duality is a local field redefinition of the scalar and vierbein - leaves S-matrix invariant!!!!!