Perturbations and GWs of black holes in quadratic gravity

NEB-21

Georgios Antoniou

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Based on Phys. Rev. D **111** (2025) 6, 064059 and an upcoming work with L. Gualtieri and P. Pani.

Introduction

What is Quadratic Gravity?

- Extension of GR with 2nd-order curvature terms.
- It is power-counting renormalizable
- Based on the action:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2 \right) \tag{1}$$

where the Weyl tensor is defined as

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = 2R_{\mu\nu} R^{\mu\nu} - \frac{2}{3} R^2 + \mathscr{G} \tag{2}$$

and the GB invariant is given by

$$\mathscr{G} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \tag{3}$$

Degrees of freedom

- the usual massless spin-2 mode $g_{\mu\nu}$

Quadratic gravity

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- massive spin-0 mode with mass $m_0^2 = 1/(6\beta)$

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- the usual massless spin-2 mode $g_{\mu\nu}$
- massive spin-0 mode with mass $m_0^2 = 1/(6\beta)$
- massive spin-2 mode with mass $\mu^2 = 1/(2\alpha)$

  - It comes with the "wrong" sign of the kinetic term $\Rightarrow$ *ghost-like*

  - This is issue is avoided in an EFT approach. However quadratic corrections can be "eliminated" by field redefinitions ($g_{\mu\nu} \rightarrow g_{\mu\nu} + c_1 g_{\mu\nu} R + c_2 R_{\mu\nu}$) [Endlich et al., 2017]

▶ It predicts hairy BHs [Lu et al., 2015]

▶ It has a well-posed IV formulation [Noakes, 1983] and numerical simulations have been performed [Held et al., 2023, 2025]

Re-writting the Lagrangian

- Static, asymptotically flat BHs in QG have $R = 0$. Without loss of generality we may set $\beta = 0$
- We introduce an auxiliary field

$$f_{\mu\nu} = -\frac{1}{\mu^2}\left(R_{\mu\nu} - \frac{1}{6}R\,g_{\mu\nu}\right) \tag{4}$$

- The action is now

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[R + 2f_{\mu\nu}G^{\mu\nu} + \mu^2\left(f_{\mu\nu}f^{\mu\nu} - f^2\right)\right] \tag{5}$$

Equations of motion

Varying with respect to $f_{\mu\nu}$

$$\mathcal{E}^{(f)}_{\mu\nu} \equiv G_{\mu\nu} + \mu^2 \left(f_{\mu\nu} - f g_{\mu\nu} \right) = 0 \tag{6}$$

Taking the covariant derivative of the above we find

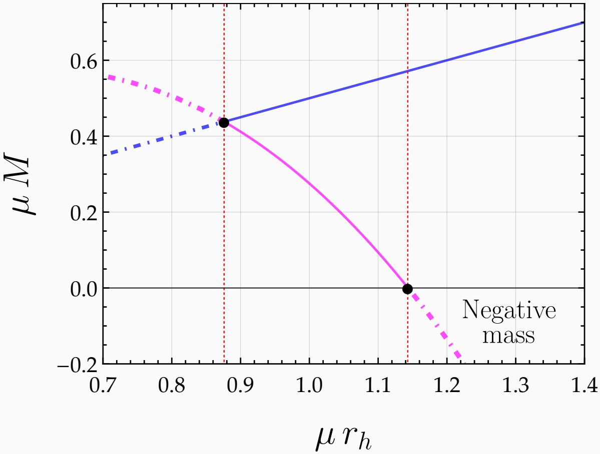
$$\mathcal{E}^{(c)}_{\mu} \equiv \nabla^{\nu} f_{\mu\nu} - \nabla_{\mu} f = 0 \tag{7}$$

Varying with respect to $g_{\mu\nu}$ we get

$$\mathcal{E}^{(g)}_{\mu\nu} \equiv \Box f_{\mu\nu} - \nabla_{\mu}\nabla_{\nu} f + 2R_{\rho\mu\sigma\nu} f^{\rho\sigma} + \mu^2 \left[f_{\mu\nu}(f-1) + g_{\mu\nu}\left(f + \frac{1}{2} f^{\alpha\beta} f_{\alpha\beta} \right) \right] = 0 \tag{8}$$

Non-GR BHs

Non-GR BHs



- The branch of non-GR BH solutions can be found for

$$0.876 \lesssim p \lesssim 1.143\,, \tag{9}$$

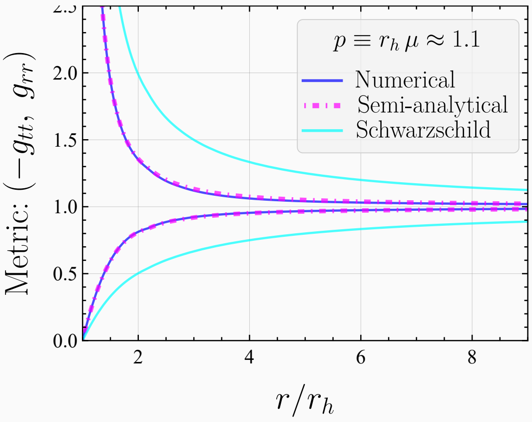
where $p \equiv r_h/\sqrt{2\alpha} \equiv r_h\mu$

- The straight line, corresponds to the Schwarzschild branch

Semi-analytic solutions

Semi-analytical solutions in spherical symmetry [Kokkotas et al., 2017]:

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$



$$A(r) \equiv x\,f(x) \quad , \quad \frac{A(r)}{B(r)} \equiv h(x)^2\,,$$

where

$$f(x) = 1 - \epsilon(1-x) - \epsilon(1-x)^2 + \tilde{f}(x)(1-x)^3\,,$$
$$h(x) = 1 + \tilde{h}(x)(1-x)^2\,,$$
$$\tilde{f}(x) = \frac{\tilde{f}_1}{1+\frac{\tilde{f}_2 x}{1+\frac{\tilde{f}_3 x}{1+\frac{\tilde{f}_4 x}{1+\ldots}}}}\,,\quad \tilde{h}(x) = \frac{b_1}{1+\frac{\tilde{h}_2 x}{1+\frac{\tilde{h}_3 x}{1+\frac{\tilde{h}_4 x}{1+\ldots}}}}\,,$$

Perturbations

Perturbations

We introduce the following perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon\, \delta g_{\mu\nu} \,, \tag{10}$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \varepsilon\, \delta f_{\mu\nu} \,. \tag{11}$$

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The decomposition now takes into account the RW gauge only for $\delta g_{\mu\nu}$

$$\delta g^{\mathrm{ax}}_{\ell m} = \begin{pmatrix} 0 & 0 & -h_0 \csc\theta\, \partial_\varphi & h_0 \sin\theta\, \partial_\theta \\ 0 & 0 & -h_1 \csc\theta\, \partial_\varphi & h_1 \sin\theta\, \partial_\theta \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} Y_{\ell m} \tag{12}$$

$$\delta f^{\mathrm{ax}}_{\ell m} = \begin{pmatrix} 0 & 0 & F_0 \csc\theta\, \partial_\phi & -F_0 \sin\theta\, \partial_\theta \\ 0 & 0 & F_1 \csc\theta\, \partial_\phi & -F_1 \sin\theta\, \partial_\theta \\ * & * & -F_2 \csc\theta\, X & F_2 \sin\theta\, W \\ * & * & * & F_2 \sin\theta\, X \end{pmatrix} Y_{\ell m} \tag{13}$$

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We distinguish between two different cases

- **Ricci tensor flat solutions:** with $\bar{R}_{\mu\nu} = 0$ (i.e. Schwarzschild background)
- **Ricci scalar flat solutions:** with $\bar{R} = 0$ (includes the hairy solutions)

Ricci-tensor flat background

The perturbation equations are simplified significantly and decouple:

$$\delta\mathcal{E}^{(g)}_{\mu\nu} = \delta G_{\mu\nu} + \mu^2 \delta f_{\mu\nu} = 0 \tag{12}$$

$$\delta\mathcal{E}^{(f)}_{\mu\nu} = \bar{\Box}\delta f_{\mu\nu} + 2\bar{R}_{\mu\sigma\nu\rho}\delta f^{\sigma\rho} - \mu^2 \delta f_{\mu\nu} = 0 \tag{13}$$

We may write the system of eqs as

$$\frac{d^2}{dr^2}\boldsymbol{\Psi} + \boldsymbol{P}\frac{d}{dr}\boldsymbol{\Psi} + \boldsymbol{V}\Psi = 0 \tag{14}$$

where $\boldsymbol{\Psi} = (\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}) \equiv (h_1, F_1, F_2)$,

Ricci-tensor flat background

We solve the system

$$\frac{d^2}{dr^2}\boldsymbol{\Psi} + \boldsymbol{P}\frac{d}{dr}\boldsymbol{\Psi} + \boldsymbol{V}\Psi = 0 \tag{15}$$

where

$$\boldsymbol{P} = \begin{pmatrix} P_{11} & 0 & P_{13} \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} V_{11} & V_{23} & V_{13} \\ 0 & V_{22} & V_{23} \\ 0 & V_{32} & V_{33} \end{pmatrix}, \tag{16}$$

Ricci-tensor flat background

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with appropriate boundary conditions

- purely outgoing modes at infinity
- purely ingoing modes at the horizon

Perturbations: Boundary conditions

Boundary conditions

$$h_1(r) = \frac{e^{-i\omega r}}{(r-2M)^{2iM\omega}} \sum_{n=0} \tilde{h}_1^{(n)} (r-2M)^{n-1}$$

$$F_1(r) = \frac{e^{-i\omega r}}{(r-2M)^{2iM\omega}} \sum_{n=0} f_1^{(n)} (r-2M)^{n-1}$$

$$F_2(r) = \frac{e^{-i\omega r}}{(r-2M)^{2iM\omega}} \sum_{n=0} f_2^{(n)} (r-2M)^{n}$$

Boundary conditions

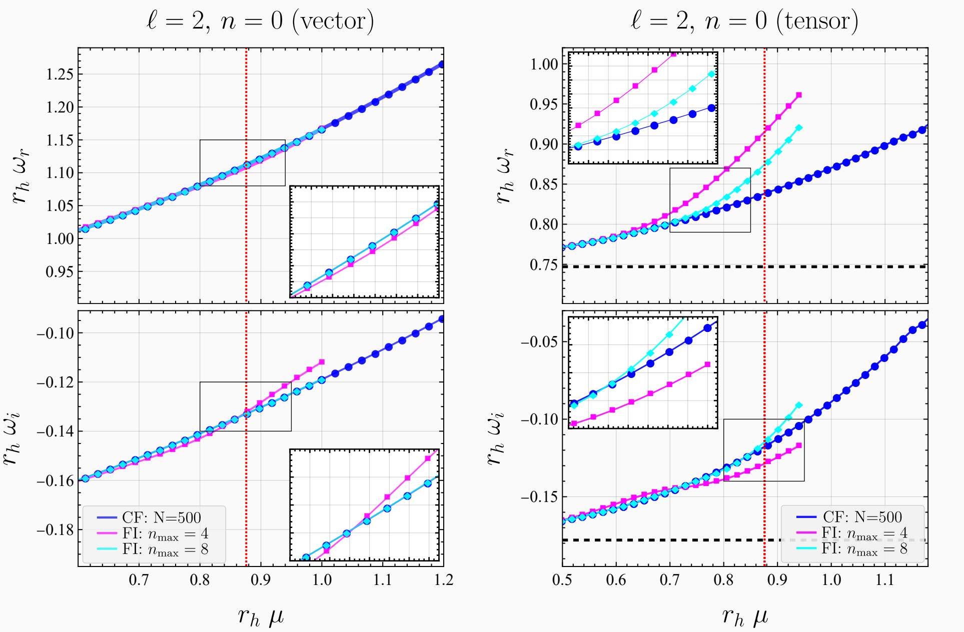
$$h_1(r) = e^{ikr} r^x \sum_{n=0} \frac{H_{1a}^{(n)}}{r^{n-1}} + e^{i\omega r} r^{2iM\omega} \sum_{n=0} \frac{H_{1b}^{(n)}}{r^{n-1}}$$

$$F_1(r) = e^{ikr} r^x \sum_{n=0} \frac{F_1^{(n)}}{r^n}$$

$$F_2(r) = e^{ikr} r^x \sum_{n=0} \frac{F_2^{(n)}}{r^{n-1}}$$

where $k = \sqrt{\omega^2 - \mu^2}$ and $x = M(\mu^2 - 2\omega^2)/(ik)$

Perturbations: Ricci-tensor flat



Perturbations: Ricci-scalar flat

Ricci-scalar flat background

The perturbation equations are now:

$$\delta G_{\mu\nu} + \mu^2 \delta f_{\mu\nu} - \mu^2 \bar{g}_{\mu\nu} \delta f = 0 \tag{17}$$

$$\begin{aligned} &\bar{\Box} \delta f_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \delta f + 2(\bar{R}_{\rho\mu\sigma\nu} \delta f^{\rho\sigma} + \delta R_{\rho\mu\sigma\nu} \bar{f}^{\rho\sigma}) + \mu^2 \Big[- \delta f_{\mu\nu} \\ &+ (\bar{g}_{\mu\nu} + \bar{f}_{\mu\nu}) \delta f + \frac{1}{2} \bar{f}^{\rho\sigma} \bar{f}_{\rho\sigma} + \bar{g}_{\mu\nu} \bar{f}_{\rho\sigma} \delta f^{\rho\sigma} \Big] = 0 \end{aligned} \tag{18}$$

We may write the system of eqs as

$$\frac{d^2}{dr^2} \boldsymbol{\Psi} + \boldsymbol{P^h} \frac{d}{dr} \boldsymbol{\Psi} + \boldsymbol{V^h} \Psi = 0 \tag{19}$$

where $\boldsymbol{\Psi} = (\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}) \equiv (h_1, F_1, F_2)$,

Ricci-scalar flat background

We solve the system

$$\frac{d^2}{dr^2}\boldsymbol{\Psi} + \boldsymbol{P^h}\frac{d}{dr}\boldsymbol{\Psi} + \boldsymbol{V^h}\Psi = 0 \tag{20}$$

where

$$\boldsymbol{P^h} = \begin{pmatrix} P^h_{11} & 0 & P^h_{13} \\ P^h_{21} & P^h_{22} & P^h_{23} \\ P^h_{31} & 0 & P^h_{33} \end{pmatrix}, \quad \boldsymbol{V^h} = \begin{pmatrix} V^h_{11} & V^h_{12} & V^h_{13} \\ V^h_{21} & V^h_{22} & V^h_{23} \\ V^h_{31} & V^h_{23} & V^h_{33} \end{pmatrix}, \tag{21}$$

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Now the perturbations are coupled $\Rightarrow$ the boundary conditions need to take that into account:

- All perturbation functions receive contributions both from the massive and massless modes at the horizon and infinity
- Again we take purely ingoing/outgoing modes at the horizon/infiinty

Perturbations: Boundary conditions

Horizon

$$h_1(r) = \frac{e^{-i\omega r}}{(r-r_h)^{i\omega/\sqrt{c\,b_1}}} \sum_{n=0} h_1^{(n)} (r-r_h)^{n-1}$$

$$F_1(r) = \frac{e^{-i\omega r}}{(r-r_h)^{i\omega/\sqrt{c\,b_1}}} \sum_{n=0} f_1^{(n)} (r-r_h)^{n-1}$$

$$F_2(r) = \frac{e^{-i\omega r}}{(r-r_h)^{i\omega/\sqrt{c\,b_1}}} \sum_{n=0} f_2^{(n)} (r-r_h)^{n}$$

Infinity

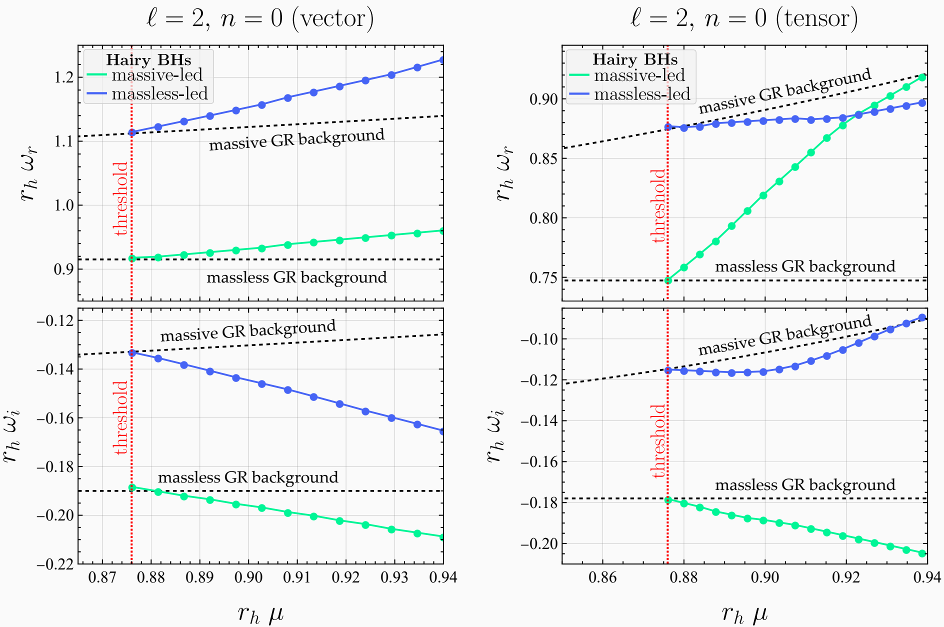
$$h_1(r) = e^{ikr} r^{x} \sum_{n=0} \frac{H_{1a}^{(n)}}{r^{n-1}} + e^{i\omega r} r^{2iM\omega} \sum_{n=0} \frac{H_{1b}^{(n)}}{r^{n-1}}$$

$$F_1(r) = e^{ikr} r^{x} \sum_{n=0} \frac{F_{1a}^{(n)}}{r^{n}} + e^{i\omega r} r^{2iM\omega} \sum_{n=0} \frac{F_{1b}^{(n)}}{r^{n}}$$

$$F_2(r) = e^{ikr} r^{x} \sum_{n=0} \frac{F_{2a}^{(n)}}{r^{n-1}} + e^{i\omega r} r^{2iM\omega} \sum_{n=0} \frac{F_{2b}^{(n)}}{r^{n-1}}$$

where c and b_1 are the coefficients appearing in the near-horizon expansions of the background solutions, $k = \sqrt{\omega^2 - \mu^2}$ and $x = M(\mu^2 - 2\omega^2)/(ik)$

Perturbations: Ricci-scalar flat



GW Emission

Mode excitation

Are the massive modes excited?

Are the massive modes excited?

To study this problem we consider a non-Trivial EM tensor which modifies the equations. For the GR background the equations read

$$\delta G_{\mu\nu} + \mu^2 \delta f_{\mu\nu} = \frac{8\pi}{3} g_{\mu\nu} T \tag{22}$$

$$\bar{\Box}\delta f_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu}\delta f^{\rho\sigma} - \mu^2 \delta f_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T + \frac{1}{3\mu^2} \bar{\nabla}_\mu \bar{\nabla}_\mu T \right) \tag{23}$$

$$\bar{\nabla}^\nu \delta f_{\nu\mu} = \frac{8\pi}{3\mu^2} \bar{\nabla}_\mu T \tag{24}$$

$$\delta f = \frac{8\pi}{3\mu^2} T \tag{25}$$

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$$\bar{\nabla}^\nu \delta f_{\nu\mu} = \frac{8\pi}{3\mu^2} \bar{\nabla}_\mu T \tag{24}$$

$$\delta f = \frac{8\pi}{3\mu^2} T \tag{25}$$

What happens in the limits?

- when $M\mu \gg 1$ do we recover GR exactly?
- when $M\mu \sim 1$?

Mode excitation

We now need to examine the polar sector

$$\delta g^{\ell m}_{\rm pol} = \begin{pmatrix} A H_0^{lm} Y^{\ell m} & H_1^{\ell m} Y^{\ell m} & 0 & 0 \\ * & B^{-1} H_2^{\ell m} Y^{\ell m} & 0 & 0 \\ 0 & 0 & r^2 H^{\ell m} Y^{\ell m} & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta H^{\ell m} Y^{\ell m} \end{pmatrix}, \tag{26}$$

$$\delta f^{\ell m}_{\rm pol} = \begin{pmatrix} A F_0^{lm} Y^{\ell m} & F_1^{\ell m} Y^{\ell m} & \eta_0^{\ell m} \partial_\theta Y^{\ell m} & \eta_0^{\ell m} \partial_\phi Y^{\ell m} \\ * & B^{-1} F_2^{\ell m} Y^{\ell m} & \eta_1^{\ell m} \partial_\theta Y^{\ell m} & \eta_1^{\ell m} \partial_\phi Y^{\ell m} \\ * & * & r^2 [K^{\ell m} Y^{\ell m} + G^{\ell m} W^{\ell m}] & r^2 G^{\ell m} X^{\ell m} \\ * & * & * & r^2 \sin^2\theta [K^{\ell m} Y^{\ell m} - G^{\ell m} W^{\ell m}] \end{pmatrix}, \tag{27}$$

Radially infalling particle: monopole

The equations we need to solve in presence of a source

$$\left(\frac{d^2}{dr^2} + \boldsymbol{P}\frac{d}{dr} + \boldsymbol{V}\right)\Psi_{\ell m} = \boldsymbol{S}_{\ell m} \qquad (28)$$

where the source matrix is defined by $\boldsymbol{S}_{\ell m}^{\top} = \left(S_{\ell m}^{(1)}, S_{\ell m}^{(2)}, S_{\ell m}^{(3)}, S_{\ell m}^{(4)}\right) \equiv (S_{\ell m}^{(K)}, S_{1\ell m}^{(\eta_1)}, S_{\ell m}^{(G)}, S_{\ell m}^{(H)})$.

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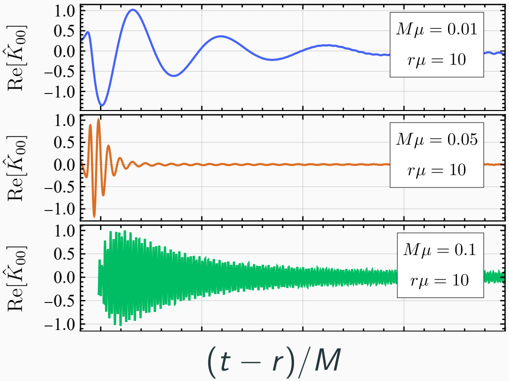
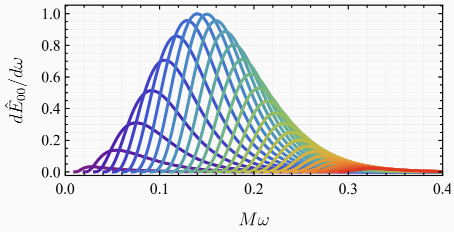
▶ For the monopole we can write a single master equation

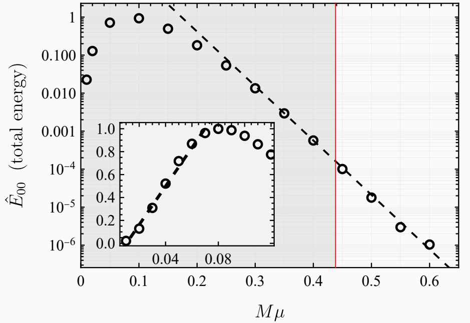
$$\frac{d^2 K^\star}{dr_\star^2} + \left(\omega^2 - V_{K^\star}\right) K^\star = S_{K^\star} \qquad (29)$$



Radially infalling particle: monopole

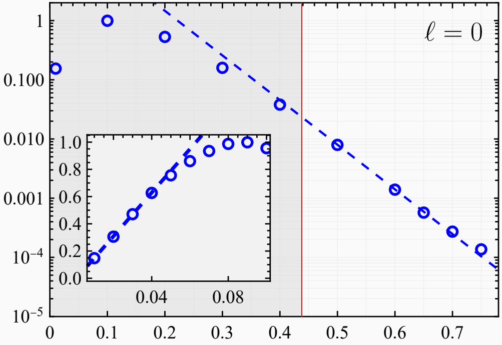
- ▶ Tracking either the waveform peaks or the amplitude we deduce an exponential suppression for masses larger than $M\mu \sim 0.1$.
- ▶ Small values of $M\mu$ have already been shown to give very small deviations from GR in massive gravity.
- ▶ This points towards extremely small deviations from GR in the regime of interest concerning masses larger than $M\mu \sim 0.45$.

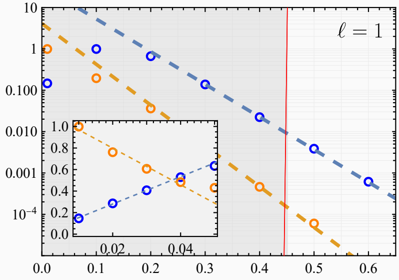


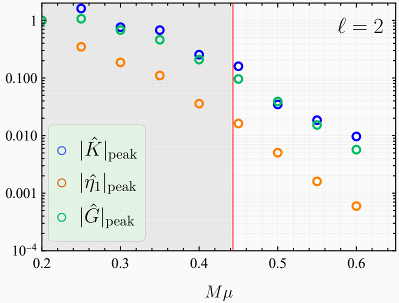


Radially infalling particle: dipole and quadrapole

- ▶ The peak of the waveforms is exponentially suppressed for any angular number ℓ





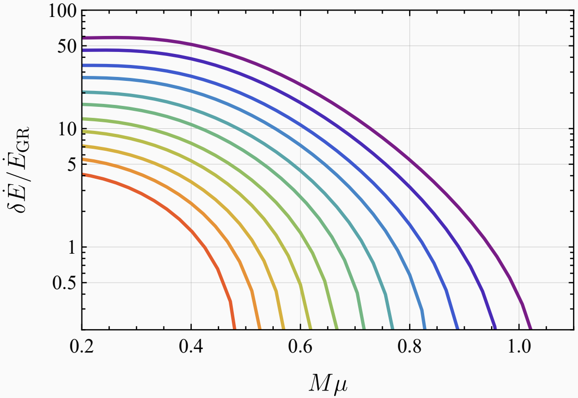


Circular orbits

- in this case we can consider the emission of the usual massless mode. The massive ones are not relevant here because the ISCO sets an upper bound on ω $\rightarrow$ we cannot probe $M\mu \gg 1$

- We focus on the $(2,2)$ mode

- Once again the GR deviations are highly suppressed in the stable regime



Outlook

- The existence of GR and hairy BHs within the framework of quadratic gravity leads to a much more complicated QNM spectrum.

- The hairy branch seems to be stable in the range where hairy solutions exist.

- In GR backgrounds, for radially plunging particles and circular orbits we deduce an exponential suppression of QG deviations from GR in the non-perturbative regime.

- Do these results translate to the hairy branch of solutions as well?

Thank you!