From Gravitating Skyrmions to QCD at finite density and Back

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NEB-21, Corfú, 1-4 September 2025

This talk is based on the following works:

1) E. Ayon-Beato, F.C., J. Zanelli, Phys. Lett. B 752 (2016), 201-205
2) P. Alvarez, F.C., N. Dimakis, A. Paliathanasis, Phys. Lett. B 773 (2017), 401-407
3) F.C. Eur. Phys. J. C (2018) 78:929
4) F.C., C. Corral, JHEP 11 (2023) 146
5) F.C., C. Corral, B. Diez, PRD 111 (2025), 084072

\+ Several other references

Very many acknowledgements!!
to all the friends, colleagues, students which have been together with me in this Journey:

E. Ayon-Beato, H. Maeda, J. Zanelli, F. Correa,
A. Giacomini, A. Paliathanasis, N. Dimakis,
A. Vera, M. Lagos, D. Hidalgo, M. Torres, S. Rebolledo,
P. Alvarez, N. Grandi, S. Cacciatori, F. Muscolino,
E. Delgado, S. Bolognesi, C. Corral, B. Diez, ...

many thanks also to Fondecyt/ANID/Exploración

Organition of the talk

1) Importance of Skyrme theory, Einstein-Skyrme theory and Gravitating solitons in general
2) the first analytic example of gravitating Skyrmions and its remarkable properties
3) Why the Skyrme field equations alone are more difficult than the Einstein-Skyrme field equations
4) from Gravitating Skyrmions to the first analytic Skyrmions on flat spaces with and without M_I
5) Back to gravitating solitons in the Euclidean case: squashed spheres in Einstein-Yang Mills and Wormholes in Einstein-Skyrme
6) Conclusions and perspectives

The Importance of Einstein-Skyrme Theory

- Skyrme theory (introduced in 1960-1961 by T. Skyrme) describe very well the low energy limit of QCD

$$S_{SK} = \frac{K}{2}\int \sqrt{-g}\, d^4x \left\{\frac{1}{2}\mathrm{tr}\left(U^{-1}\partial_\mu U\right)^2 + \frac{\lambda}{16}\mathrm{tr}\left(\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2\right)\right\} + \text{subleading terms}$$

- Pions mass is negligible in several situations involving heavy solitons
- Other Subleading terms are $O(1/N_c)$ in 't Hooft expansion: in ~1982-84 Witten, Balachandran et al, Adkins, Nappi, ... Skyrme is Low energy Limit of QCD
- The Skyrme term is added to the NonLinear Sigma model in order to avoid Derrick Scaling arguments

We will see explicitly in the following that, indeed, the Skyrme term helps to resist against the Gravitational attraction

We have two Coupling constants (K, λ) which can be determined from experiments. In principle, these coupling can be computed with QCD

$U \in SU(N_f)$, N_f = number of flavors.
Here we will mainly consider $N_f = 2$ in which case the theory describes the 3 Pions, Protons, Neutrons

Thus, U is a scalar field which takes Value in $SU(2)$. The internal symmetry of the Theory is the Isospin symmetry of Nuclear physics

Remarkable idea by T. Skyrme

the trivial vacuum of the theory is

$U_0 = \mathbb{1}_{2\times2}$, The small excitations around the vacuum $U = U_0 + \delta U$, $|\delta U| \ll 1$, $|\partial_\mu \delta U| \ll 1$

represents Pionic excitations (which are Bosonic).
However, Skyrme had the idea that the Solitons of a Bosonic theory can behave as Fermions and, in particular, the Solitons of Skyrme theory describe Baryons (Protons and neutrons in the $SU(2)$ case)

Field Equations of Skyrme theory

$$\text{F.E.:}\ \nabla^\mu\left(U^{-1}\partial_\mu U + \frac{\lambda}{4}\left[U^{-1}\partial^\nu U, \left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]\right]\right) = 0$$

It is fair to say that this system of 3 coupled nonlinear PDE is the prototype of non-integrable field Theory

Not surprising as the Skyrme model describes QCD in the Strong Coupling limit

Topological charge:

$$Q = B = -\frac{1}{24\pi^2}\int_\Sigma \epsilon^{ijk}\,\mathrm{tr}\left\{\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right\}$$

the topological charge is interpreted here as the Baryonic charge of the Configuration and ρ_B as the Baryonic charge density

$$\rho_B = -\frac{1}{24\pi^2}\epsilon^{ijk}\,\mathrm{tr}\left\{\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right\}$$

Hence, we would like to find analytic solutions of the field equations with $\rho_B \neq 0$

Unfortunately, Standard methods fail: for instance, the BPS bound in the Skyrme case cannot be Saturated

Therefore, New Techniques must be introduced to solve Skyrme field equations on flat spaces!

Energy for Static configurations:

$$\mathcal{E} = \int d^3x \left\{\frac{1}{2}\mathrm{tr}\left[\left(U^{-1}\partial_i U\right)^2 + \frac{\lambda}{4}\left[U^{-1}\partial_i U, U^{-1}\partial_j U\right]^2\right]\right\}$$

$$\Rightarrow\ \mathcal{E} = \int d^3x \left\{\frac{1}{2}\mathrm{tr}\left[\left(U^{-1}\partial_i U \pm \tilde{c}_1\epsilon_{ijk}\left[U^{-1}\partial_j U, U^{-1}\partial_k U\right]\right)^2\right] \mp \tilde{c}_2\rho_B\right\} \Rightarrow \mathcal{E} \geq \tilde{c}_3|B|$$

$\Downarrow$

However, unlike what happens for superconducting Vortices at critical coupling, non-Abelian monopoles, self-dual Instantons... the BPS equations $U^{-1}\partial_i U = \pm\tilde{c}_1\epsilon_{ijk}\left[U^{-1}\partial_j U, U^{-1}\partial_k U\right]$ only have trivial solutions on flat spaces

Einstein - Skyrme theory

Einstein - Skyrme theory describes the gravitational effects of the Low energy limit of QCD $\longrightarrow$ Thus, it is a very relevant theory

Moreover, as the Skyrmions behaves as a Fermions it avoids the technical problems to describe gravitating Fermions with Grassman Variables coupled to general Relativity ...

Quite Remarkably, in a sense, the field equations of the coupled Einstein - Skyrme system are easier

First example of gravitating Skyrmion

$$I[g,U] = \int d^4x \sqrt{-g} \left\{ \frac{R - 2\Lambda}{2\bar{\kappa}} - \frac{K}{4} \mathrm{tr}\left[(U^{-1}\partial_\mu U)^2 + \frac{\lambda}{8} [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]^2 \right] \right\}$$

Field equations

$$\begin{cases} \nabla^\mu \left(U^{-1}\partial_\mu U + \frac{\lambda}{4} [U^{-1}\nabla^\nu U, [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]] \right) = 0 \\[2ex] G_{\mu\nu} + \Lambda g_{\mu\nu} = \bar{\kappa} T_{\mu\nu} \end{cases}$$

where $\bar{\kappa}$ is the Newton Constant, Λ is the Cosmological constant and

$$T_{\mu\nu} = -\frac{K}{2} \mathrm{tr}\Big\{ (U^{-1}\partial_\mu U)(U^{-1}\partial_\nu U) - \frac{1}{2} g_{\mu\nu} (U^{-1}\partial_\sigma U)(U^{-1}\nabla^\sigma U)$$

$$+ \frac{\lambda}{4} \Big[g^{\alpha\beta} [U^{-1}\partial_\mu U, U^{-1}\partial_\alpha U][U^{-1}\partial_\nu U, U^{-1}\partial_\beta U] - \frac{g_{\mu\nu}}{4} [U^{-1}\partial_\alpha U, U^{-1}\partial_\beta U]^2 \Big] \Big\}$$

We want to find an analytic solution of the above system with unit Baryonic charge: $B = 1$

$\boxed{\text{It looks a hopeless task}}$ however let us consider the following Ansatz

indeed, on flat spacetime there are no analytic solutions with $B=1$ and spherical symmetry

it easy to see that in this area the most important and artistic skill is to construct good ansatze

Matter field Ansatz

$$U = \cos\alpha \, \mathbb{I}_{2\times 2} + \sin\alpha \, n_I t^I \; ; \; \vec{n} = (\sin F \cos G, \sin F \sin G, \cos F)$$

where ($\alpha = \alpha(x^\mu)$, $F = F(x^\mu)$, $G = G(x^\mu)$) are the 3 scalar degrees of freedom of the $SU(2)$ Skyrme model

$$G = \frac{\gamma + \varphi}{2}, \quad \tan F = \frac{\cot\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\gamma - \varphi}{2}\right)}, \quad \tan\alpha = \frac{\sqrt{1 + \tan^2 F}}{\tan\left(\frac{\gamma - \varphi}{2}\right)}$$

Metric Ansatz

$$ds^2 = -dt^2 + \frac{\rho_0^2}{4} \Big[\underbrace{(d\gamma + \cos\theta \, d\varphi)^2 + d\theta^2 + \sin^2\theta \, d\varphi^2}_{\text{this is the metric of a 3-sphere}} \Big]$$

Quite Remarkably, the Skyrme field equations are identically satisfied with the above ansatz for $U \in SU(2)$ in the above metric

Thus, the coupled field equations of the Einstein - Skyrme system Reduce to:

$$\boxed{\rho_0^2 = \frac{3(2 - K\bar{\kappa})}{4\Lambda}, \quad \lambda = \frac{3(2 - K\bar{\kappa})^2}{8\Lambda K\bar{\kappa}}}$$

Such exact solution has $B = 1$ So, it represents a gravitating Skyrmion with unit Baryonic charge

Such solution can be further generalized:

Bianchi - $\underline{IX}$ ansatz for the metric

$$ds^2 = -dt^2 + \frac{\rho^2(t)}{4} \left[a^2(t) (d\gamma + \cos\theta \, d\varphi)^2 + d\theta^2 + \sin^2\theta \, d\varphi^2 \right]$$

While, for $SU(2)$-Valued matter field we keep exactly the same ansatz

Quite Remarkably, despite the fact that the metric ansatz is more general, the same ansatz for the Skyrme field works in exactly the same way in the above Bianchi $\underline{IX}$ metric

$\rightarrow$ namely, the Skyrme field equations are identically satisfied in the sector with unit Topological charge

Note that on flat spaces one would pay gold coins to get ansatz for Skyrme such that the field equations are identically satisfied!!

Thus, with the above Bianchi $\underline{IX}$ ansatz for the metric and with the original ansatz for the Skyrme field the full Einstein - Skyrme System reduces (in the case $a(t) = 1$) to

$$\begin{cases} \dot{\rho}^2 = \frac{\Lambda}{3}\rho^2 + \frac{\lambda K \bar{\kappa}}{2\rho^2} + \frac{K\bar{\kappa} - 2}{2} \\[2ex] \ddot{\rho} = \frac{\Lambda}{3}\rho - \frac{\lambda K \bar{\kappa}}{2\rho^3} \end{cases}$$

One can solve analytically the above system and Thanks to the presence of the Skyrme coupling ($\lambda > 0$) Cosmological Bounces with $\rho(t) \neq 0 \; \forall t$ are possible ...

But the question is: Why the Skyrme field equations are identically satisfied on the above family of metric?

This is very strange indeed as, usually, the Skyrme field equations are the difficult part of the problem ...

Answer:

The ansatz for the Skyrmion has the following property:

$$\underbrace{U^{-1}\partial_\mu U}_{\text{Maurer - Cartan}} = \sum \Omega_\mu^a \tau_a$$ where the Ω_μ^a are the left-invariant form of U

It turns out that the spatial Vierbein of the Bianchi $\underline{IX}$ are exactly the left-invariant forms associated to the matter field

$$\underbrace{\Omega_\mu^a \, dx^\mu}_{\text{Matter field}} = \underbrace{e^a}_{\text{Vierbein of the metric}}$$ where $ds^2 = -dt^2 + \sum_{a=1}^{3} I_a^2(t) \, e^a \otimes e^a$

for this reason, when one computes the Skyrme field equations, one has to contract Vierbein with inverse Vierbein and then take derivatives of these contractions which always vanish $\Rightarrow$ the Skyrme field equations are identically satisfied!!

How these results help to construct the first analytic skyrmions on flat spaces

The previous results clarify why the Einstein Skyrme system can be easier than the Skyrme system alone $\Rightarrow$ Build the metric using the left-invariant form arising from the matter field

Note that this strategy is also very useful in Einstein-Yang-Mills theory

Unfortunately, this strategy does not work on flat spaces ... But

Useful Characteristics of the SU(2) ansatz

Let us give a closer look at the ansatz for $U \in SU(2)$:

$$
* \left\{
\begin{aligned}
&U = \cos\alpha \, \mathbb{1}_{2\times 2} + \sin\alpha \, n_j t^j \\
&\vec{n} = (\sin F \cos G, \sin F \sin G, \cos F) \\
&G = \frac{\gamma+\varphi}{2} \; ; \quad \tan F = \frac{\cot\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\gamma-\varphi}{2}\right)} \; ; \quad \tan\alpha = \frac{\sqrt{1+\tan^2 F}}{\tan\left(\frac{\gamma-\varphi}{2}\right)} \; ;
\end{aligned}
\right.
$$

Relevant Characteristics

In order to export the nice properties of the above ansatz to the case of skyrmions in flat spaces let us list the most useful characteristics of $*$:

It is useful to rewrite as

$$
*_2 \left\{
\begin{aligned}
&G = \frac{\gamma+\varphi}{2} \; ; \quad \tan F = \frac{\tan H}{\cos A} \; ; \quad \tan\alpha = \frac{\sqrt{1+\tan^2 F}}{\tan A} \; ; \\
&A = \frac{\gamma-\varphi}{2} \; ; \quad \tan H = \cot\left(\frac{\theta}{2}\right) \; ;
\end{aligned}
\right.
$$

One of the most important characteristics of $*_2$ is that it is constructed in such a way that

$$
*_3 \left\{ \nabla_\mu G \nabla^\mu A = 0 = \nabla_\mu G \nabla^\mu H = \nabla_\mu H \nabla^\mu A \right.
$$

If the above $*_3$ hold, then the expressions for the action and energy-momentum tensor simplify considerably ...

Idea !!

Let us use a similar Ansatz but on flat spaces!

Note that the usual spherical Hedgehog for Skyrme (on textbooks) does not have this form

$$
\text{Flat} \left\{
\begin{aligned}
&U = \cos\alpha \, \mathbb{1}_{2\times 2} + \sin\alpha \, n_j t^j , \\
&\vec{n} = (\sin F \cos G, \sin F \sin G, \cos F) \\
&G = \frac{\gamma+\phi}{2} , \quad \tan F = \frac{\tan H}{\cos A} , \quad \tan\alpha = \frac{\sqrt{1+\tan^2 F}}{\tan A} \\
&A = \frac{\gamma-\phi}{2} , \quad H = H(t,x)
\end{aligned}
\right.
$$

Note that in order to keep alive the topological charge density the 3 functions (G, A, H) must be independent: $dG \wedge dA \wedge dH \neq 0$

$$ds^2 = -dt^2 + \ell^2 (dx^2 + d\gamma^2 + d\phi^2)$$

Here, (x, γ, ϕ) are flat spatial a-dimensional coordinates and t is the time, ℓ has dimension of length

$\to$ this is the simplest flat metric where the properties of the ansatz are kept alive

The importance of the parameter ℓ will be clear in a moment

Remarkable features of the ansatz

1) The topological (Baryon) charge density is non-vanishing: $\rho_B = 3 \sin(2H) \, dH \wedge d\gamma \wedge d\phi$

2) The field equations reduce consistently to just one PDE for $H(t,x)$: Note that this is a quite non-trivial property.

Indeed, in the ansatz there is only one free function (as the rest of the dependence of $U \in SU(2)$ on the space-time coordinates has been fixed as in the case of the gravitating skyrmion)

While the $SU(2)$ Skyrme field equations are 3 coupled nonlinear PDE: $0 = \nabla^\mu \left(U^{-1}\partial_\mu U + \frac{1}{4}\left[U^{-1}\partial^\nu U, \left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]\right] \right)$

$\to$ We have as many equations as the dimension of the Lie Algebra of the Isospin group (3 in the $SU(2)$ case) and we have 1 free function to satisfy 3 equations ..

The Most Remarkable property

The 3 field equations reduce to

$$\boxed{\Box H - \frac{\lambda}{8\ell^2(\lambda + 2\ell^2)} \sin(4H) = 0}$$

In other words, the field equations of the prototype of non-integrable theories reduce (with a suitable ansatz) to the field equation of the prototype of integrable field theory: Sine-Gordon equation !!!

We do not have one but actually ∞ many solutions

Boundary Conditions

$$ds^2 = -dt^2 + \ell^2 (dx^2 + d\gamma^2 + d\phi^2)$$

Dirichlet B.C. in x: $\left\{ \begin{aligned} &H(0) = 0, \\ &H(2\pi) = \pm \frac{\pi}{2} \end{aligned} \right.$

Periodic Boundary Condition in (γ, φ)

With these B.C.: $B = \pm 1$

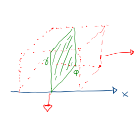
The ansatz can be further generalized:

$\left\{ \begin{aligned} &\gamma \to p\gamma \\ &\varphi \to q\varphi \end{aligned} \right. , \quad p, q \in \mathbb{N} \to B = \pm pq$

While the effective Sine-Gordon coupling read: $\dfrac{p^2 q^2 \lambda}{4\ell^2(4\ell^2 + \lambda(p^2+q^2))} = g_{eff}$

These solutions represent Baryonic layers living at finite Baryon density

Note that these Baryonic layers do not form in free space they need to be confined in a finite spatial volume



We are studying the Skyrme model in a Box

The thickness of the layer depends on the effective Sine-Gordon Coupling

These are the first analytic solutions of the Skyrme model on flat spaces with $B \neq 0$

Ordered arrays of Baryonic tubes

Using in a clever way the first analytic gravitating Skyrmions, we have constructed the first analytic solutions of the Skyrme model in flat space in (3+1)D Disclosing a remarkable mapping with Sine-Gordon theory in (1+1)D

These configurations represent Baryonic Layers whose thickness is fixed by the Baryonic charge and the coupling of the theory

Why that ansatz is so good?

Without recognizing it clearly, we were actually using Euler Angles instead of the usual exponential representation!!

$$U = \exp(t_3 F)\exp(t_2 H)\exp(t_3 G)$$

At the Beginning of our investigation we actually did not recognize that we were using the Euler-Angle representation !!

$$ds^2 = -dt^2 + dx^2 + L^2(d\gamma^2 + d\varphi^2)$$

$H = H(t,x)$, $F = \frac{q}{2}\gamma$, $G = \frac{p}{2}\varphi$ (for simplicity, take $p = q = B^{1/2}$)

$$\partial_t^2 H - \partial_x^2 H + \left(\frac{B^2\lambda}{8L^2(2L^2 + B\lambda)}\right)\sin(4H) = 0$$

$$\rho_B = -3B(\sin(2H))\partial_x H$$

We have a complete mapping with Sine-Gordon theory together with the analysis of its range of validity in: PRD 108,114017(2023)

However, the question is:

are there other solutions with different shapes?

Numerical simuations suggest that, besides Baryonic Layers, also Baryonic tubes should appear

It is worth emphasizing that numerical simulations are performed (with Big Machines) putting many Baryons in a Box interacting with a phenomenological Baryon - Baryon potential

the simulations know nothing about Skyrme theory or QCD

Box

• = Baryons

V_{BB} = .. experimental fit ..

In the Box there are (more than) 10^{10} Baryons interacting with the phenomenological potential

Need for a compromise when searching for Good Ansatz ...

On the one hand

Field equations: $0 = \nabla^\mu\left(U^{-1}\partial_\mu U + \frac{1}{4}[U^{-1}\partial^\nu U, [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]]\right)$

Very non-linear → Need to simplify

$U \in SU(2)$ as much as possible (to have some hope to solve them analytically)

On the other hand

Baryonic charge: $B = -\frac{1}{24\pi^2}\int_{\{t=const\}}\varepsilon^{ijk}\,\mathrm{tr}\left[(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)\right]$

The problem is that if one simplifies the ansatz for $U \in SU(2)$ too much then the Baryonic charge vanishes identically

For instance, if
$$\begin{cases} U = \cos\alpha\,\mathbb{I} + \sin\alpha\, n_j t^j \\ \vec{n} = (\sin F\cos G, \sin F\sin G, \cos F) \\ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ \alpha = \alpha(r),\ F = F(r),\ G = G(r) \end{cases}$$

This ansatz is spherically symmetric as U only depends on r

But the Baryonic charge vanishes identically

This 'tension' arises even before trying to solve the field equations ...

Indeed, let us just require a consistent ansatz "with spherical symmetry" such that

1) The 3 field equations reduce to just 1 equation for a suitable profile

2) The Baryonic charge is non-zero

*) Naive spherical symmetry: $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$U = \cos\alpha\,\mathbb{I} + \sin\alpha\, n_j t^j$, $\quad \alpha = \alpha(r)$, $F = F(r)$, $G = G(r)$

$\vec{n} = (\sin F\cos G, \sin F\sin G, \cos F)$

$$\mathcal{L}_{\vec{R}}\, U = 0 \longrightarrow B = 0$$

**) Spherical Hedgehog symmetry

$$\begin{cases} \text{(I)}\ \mathcal{L}_{\vec{R}}\, U \neq 0 \\ \text{(II)}\ \mathcal{L}_{\vec{R}}\, T_{\mu\nu} = 0 \end{cases}$$

⟹ thus, let us give up sperical symmetry for $U \in SU(2)$ requiring only spherical symmetry for $T_{\mu\nu}$

(I)+(II): the above two conditions fix uniquely the ansatz to be: $(\alpha(r),\ F = \theta,\ G = \varphi)$

Keeping alive at the same time the Baryonic charge density: $\rho_B \propto \sin\theta(\sin^2\alpha)\partial_r\alpha$

However, we should be worried since in the ansatz there is only one free function left ($\alpha(r)$) and we need to solve 3 Skyrme field equations

In fact (Palais theorem 'principle of Symmetric criticality')

the 3 Skyrme field equations reduce to only one ODE for the profile $\alpha(r)$:

$$\left(2 + \lambda\frac{\sin^2\alpha}{r^2}\right)\alpha'' + \cdots = 0$$

This equation must be solved numerically

Let us derive this result in a "Stupid way"

Generalized Hedgehog

Note that the Euler-Angles ansatz has a similar property. Introduce

$\mathcal{L}_{\vec{T}} = (\partial_\gamma, \partial_\varphi)$ = The Lie derivative corresponding to the traslations along (γ,φ). Then, we can require

$$\begin{cases} \left(\mathcal{L}_{\vec{T}}\, U = (\partial_\gamma U, \partial_\varphi U)\right) \neq 0 \\ \mathcal{L}_{\vec{T}}\, T_{\mu\nu} = 0 \end{cases}$$

So that U is not invariant under traslation in (γ,φ) in order to keep alive the topological charge

→ So that the energy density is the one of a Baryonic Layer

these two conditions gives rise to the "Sine-Gordon ansatz" defined above with only one free function ($H(t,x)$) Nevertheless, the 3 Skyrme field equations reduce to only one equation for $H(t,x)$

"Generalized Hedgehog Ansatz"

"The Stupid Way"

In order to check whether or not different shapes (Besides Baryonic Layers) appear let us derive the Coleman-Palais theorem in a Stupid Way

Generic $U \in SU(2) \Rightarrow 3$ scalar degrees of freedom

$$U = \cos\alpha\, \mathbb{1}_{2\times 2} + \sin\alpha\, n_j t^j, \quad \vec{n} = (\sin F \cos G, \sin F \sin G, \cos F)$$

$$\alpha = \alpha(x^\mu), \quad F = F(x^\mu), \quad G = G(x^\mu)$$

We will not assume any symmetry for (α, F, G) so that, in principle, these will depend on all the 4 space-time coordinates

The 3 Skyrme field equations in full Glory

Now I will write the 3 Skyrme field equations for (α, F, G) explicitly and we will check under which conditions reduce to only one ODE for α:

the equation from $\delta\alpha$:

$$\begin{aligned}
&\left(-\Box\alpha + \tfrac{1}{2}\sin(2\alpha)\left[(\nabla F)^2 + \sin^2 F\,(\nabla G)^2\right]\right) + \\
&+\lambda\Big\{ \frac{\sin(2\alpha)}{2}\left[(\nabla\alpha)^2(\nabla F)^2 - (\nabla\alpha\cdot\nabla F)^2\right] + \\
&+\frac{\sin(2\alpha)}{2}\sin^2 F\left[(\nabla\alpha)^2(\nabla G)^2 - (\nabla\alpha\cdot\nabla G)^2\right] + \\
&+\sin^2\alpha\,\sin(2\alpha)\,\sin^2 F\left[(\nabla F)^2(\nabla G)^2 - (\nabla F\cdot\nabla G)^2\right] + \\
&+\nabla_\mu\left[\sin^2\alpha\left((\nabla\alpha\cdot\nabla F)\nabla^\mu F - (\nabla F)^2\nabla^\mu\alpha\right)\right] + \\
&+\nabla_\mu\left[\sin^2\alpha\,\sin^2 F\left((\nabla\alpha\cdot\nabla G)\nabla^\mu G - (\nabla G)^2\nabla^\mu\alpha\right)\right]\Big\} = 0
\end{aligned}$$

the equation from δF:

$$\begin{aligned}
&-\sin^2\alpha\,\Box F - \sin(2\alpha)(\nabla\alpha\cdot\nabla F) + \frac{\sin^2\alpha}{2}\sin(2F)(\nabla G)^2 \\
&+\lambda\Big\{\frac{\sin^2\alpha}{2}\sin(2F)\left[(\nabla\alpha)^2(\nabla G)^2 - (\nabla\alpha\cdot\nabla G)^2\right] + \\
&+\frac{\sin^4\alpha}{2}\sin(2F)\left[(\nabla F)^2(\nabla G)^2 - (\nabla F\cdot\nabla G)^2\right] + \\
&+\nabla_\mu\left[\sin^2\alpha\left((\nabla\alpha\cdot\nabla F)\nabla^\mu\alpha - (\nabla\alpha)^2\nabla^\mu F\right)\right] + \\
&+\nabla_\mu\left[\sin^4\alpha\,\sin^2 F\left((\nabla F\cdot\nabla G)\nabla^\mu G - (\nabla G)^2\nabla^\mu F\right)\right]\Big\} = 0
\end{aligned}$$

the equation from δG:

$$\begin{aligned}
&-\sin^2\alpha\,\sin^2 F\,\Box G - \sin(2\alpha)\sin^2\alpha(\nabla\alpha\cdot\nabla G) + \frac{\sin^2\alpha}{2}\sin(2F)(\nabla G)^2 + \\
&+\lambda\Big\{\nabla_\mu\left[\sin^2\alpha\,\sin^2 F\left((\nabla\alpha\cdot\nabla G)\nabla^\mu\alpha - (\nabla\alpha)^2\nabla^\mu G\right)\right] + \\
&+\nabla_\mu\left[\sin^4\alpha\,\sin^2 F\left((\nabla F\cdot\nabla G)\nabla^\mu F - (\nabla F)^2\nabla^\mu G\right)\right]\Big\} = 0
\end{aligned}$$

The Stupid Way to arrive at Coleman-Palais:

We must replace in the field equations:

$$\begin{cases} ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ \alpha = \alpha(r), \quad F = \theta, \quad G = \varphi \end{cases}$$

and verify that the 3 field equations reduce to Just 1 consistent equation for $\alpha(r)$ $\longrightarrow$ Indeed this is what happens ...

General Considerations:

Many terms cancel out when

$$\nabla\alpha\cdot\nabla G = 0 = \nabla G\cdot\nabla F = \nabla\alpha\cdot\nabla F$$

and, in spherical coordinates, when $(\alpha = \alpha(r),\ F = \theta,\ G = \varphi)$ two field equations are identically satisfied and one is left with only 1 ODE for $\alpha(r)$

Check, term by term, all the possible dangerous ones:

One can see easily that if $\nabla\alpha\cdot\nabla F = 0 = \nabla G\cdot\nabla F = \nabla\alpha\cdot\nabla G$ (which Basically implies that the 3 D.O.F. depend on different coordinates) then inconsistent equations can appear.

In particular check: (from the first equation)

$$-\Box\alpha + \frac{\sin(2\alpha)}{2}\sin^2 F\,(\nabla G)^2 + \cdots$$

this term only depends on the radial coordinate

this term could spoil the consistency of the equation due to $\sin^2 F(\nabla G)^2$

Thus, all the dangerous terms contain

$$\boxed{\sin^2 F\,(\nabla G)^2 = I}$$

Fortunately, in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

and when $(F = \theta,\ G = \varphi)$ it turns out that $I = \frac{1}{r^2}$ only depends on the radial coordinate and does not spoil the consistency of the ansatz

Is this the only Possibility?

Quite Remarkably, there is another much simpler possibility if we give up spherical symmetry!!

it seems that to give up spherical symmetry was the difficult part: it is usually assumed that the spherical case is the simplest

*) We need $\rho_B \neq 0 \Rightarrow d\alpha\wedge dF\wedge dG \neq 0$

$\longrightarrow$ (α, F, G) must be independent functions

*) We need $\nabla\alpha\cdot\nabla F = \nabla\alpha\cdot\nabla G = \nabla F\cdot\nabla G = 0$ otherwise the equations are not consistent

*) We need that $\sin^2 F(\nabla_\mu G\nabla^\mu G)$ only depends on the coordinates of α

$\longrightarrow$ $\boxed{\text{take } \nabla_\mu G\nabla^\mu G = 0\ !!}$

The easiest way to satisfy the above condition is not the spherical ansatz but

$$\begin{cases} ds^2 = -dt^2 + dx^2 + L^2(d\gamma^2 + d\phi^2), \quad (\gamma, \phi)\text{- adimensional coordinates} \\ \alpha = \alpha(x), \quad F = q\gamma \\ G = pu + \Phi(u), \quad u = \frac{t}{L} - \phi \end{cases}$$

Thus, we eliminate all the dangerous terms since

$$\boxed{(\nabla G)^2 = 0,\ \nabla\alpha\cdot\nabla F = 0 = \nabla\alpha\cdot\nabla G = \nabla F\cdot\nabla G}$$

The three field equations reduce to only one equation for $\alpha(x)$ which (unlike what happens in the spherical case) can be solved analytically in term of elliptic functions

Indeed the equation for $\alpha(x)$ reduces to a quadrature:

$$0 = \partial_x\left\{(L^2 + q^2\lambda\sin^2\alpha)\frac{(\partial_x\alpha)^2}{2} - \frac{(qL)^2}{2}\sin^2\alpha - E_0\right\}$$

where E_0 is an integration constant (see EPJC (2018) 78 : 929)

Note that these configurations are much simpler than the spherical ones, despite possessing less symmetries ...

The Solutions represent ordered arrays of Baryonic tubes

Note that the energy density depends explicitly on 3 coordinates and the Baryonic density as well:

$$T_{00} = T_{00}(x, \gamma, u), \quad \rho_B = \rho_B(x, \gamma, u)$$

Note that here a solution with only one Killing vector is much simpler than a configuration with 4 Killing

Back to gravity:

the simplicity of the analytic solutions representing ordered arrays of Baryonic Tubes (which are the first analytic examples of crystal-like solutions with non-vanishing B in the Skyrme model) suggests to try a similar ansatz in the gravitating case $\longrightarrow$ See A. Giacomini's talk

Euclidean Solutions in Einstein-Skyrme and Einstein-Yang-Mills

Everything started from the observation that, in the Skyrme-Einstein case, it is very convenient to use as vielbein the same left-invariant forms of the $SU(2)$-valued matter field

try a similar strategy in Einstein-Yang-Mills

Let us apply this strategy to the following concrete problem: Can we have the squashed 3-sphere as an exact solution of Einstein-Yang-Mills theory?

the squashed 3 sphere:

$$\begin{cases} ds^2 = \rho_0^2\left(\sigma_1^{\,2} + \sigma_2^{\,2} + \alpha^2\sigma_3^{\,2}\right) \\ \sigma_1 = \cos\psi\, d\theta + \sin\theta\sin\psi\, d\varphi \\ \sigma_2 = -\sin\psi\, d\theta + \sin\theta\cos\psi\, d\varphi \\ \sigma_3 = d\psi + \cos\theta\, d\varphi \end{cases}$$

$\rightarrow$ This metric has many relevant applications:

*) Spontaneous symmetry breaking in the AdS/CFT context (where the squashing parameter $\sim$ deformation parameter)

*) Correlators of Higgs and Coulomb branch operators in $\mathcal{N}=4$ SYM

*) Spontaneous symmetry breaking and localization in path Integrals

. . . .
and so on

However, In all these papers the squashed sphere has always been considered as a fixed background. Until very recently, No one was able to produce the squashed 3-sphere as a self-consistent solution of GR minimally coupled to a sound Matter field

*) Note that the squashed sphere is neither maximally symmetric nor constant curvature

Gravitating anisotropic merons and squashed 3-spheres

We will consider the $SU(2)$ case

$$I_{EYM} = \kappa\int_M d^3x\sqrt{|g|}\,(R-2\Lambda) + I_{YM}$$

$$\kappa = \frac{1}{16\pi G}\,, \qquad I_{YM} = \frac{1}{2e^2}\int_M d^3x\sqrt{|g|}\;\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$$

Pauli matrices

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,; \quad A_\mu = A_\mu^i t_i\,, \quad t_j = -\frac{i}{2}\tau_j$$

Field Equations

$$\begin{cases} \nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0 \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{1}{2\kappa}T_{\mu\nu} \end{cases}$$

where $T_{\mu\nu} = -\frac{2}{e^2}\mathrm{tr}\left\{F_{\mu\lambda}F_\nu^{\ \lambda} - \frac{g_{\mu\nu}}{4}F_{\lambda\rho}F^{\lambda\rho}\right\}$

$\rightarrow$ the Chern-Simons term can also be included in the analysis

Meron Ansatz for YM field

$$A_\mu = A_\mu^i t_i = \sum_{j=1}^{3} c_{(j)}\,\sigma^j t_j$$

where the $c_{(j)}$ are constant different from 1:

$c_{(j)} \neq 1$.

When $c_{(j)} = 1\ \forall j \Rightarrow A_\mu$ is a pure gauge

when $c_{(j)} = 1/2\ \forall j \Rightarrow A_\mu$ is the Isotropic Meron

and the Isotropic merons prefer 3-spheres without squashing $\longrightarrow$ We need to analyze configurations with $c_{(j)} \neq 1, 1/2$

then, the results from the gravitating Skyrmions suggest to build the metric ansatz using as drei-bein the same one appearing in the construction of $A_\mu = \sum_j c_{(j)}\sigma^j t_j \Rightarrow$ metric ansatz:

$$ds^2 = \rho_0^2\left(\sigma_1^2 + \sigma_2^2 + \alpha^2\sigma_3^2\right)$$

where α is the squashing parameter while ρ_0 represents the size of the spacetime $\rightarrow$

Note that this α should not be confused with Skyrme profile

Indeed $\alpha = 1$ corresponds to the round 3-sphere

Field Equations

quite Remarkably, the complete set of Einstein-Yang-Mills field equations reduce to algebraic (instead of differential) equations for the parameters $(\rho_0, \alpha, c_{(j)})$

Solutions

$$\begin{cases} c_{(1)} = c_{(2)}\,, \\ c_{(2)} = \frac{1}{4}\left[2\alpha\left(\sqrt{9\alpha^2+16} - 3\alpha\right)\right]^{1/2} \\ c_{(3)} = \dfrac{(2+\alpha^2)\left(\sqrt{9\alpha^2+16} - 3\alpha\right)}{2\left(\sqrt{9\alpha^2+16} + \alpha\right)} \end{cases}$$

where one can see that $\frac{c_{(3)}}{c_{(1)}}$ is a measure of the meron anisotropy

$$\lim_{\alpha\to+\infty}\frac{c_{(3)}}{c_{(1)}} = \frac{1}{\sqrt{3}}$$

$\rightarrow$ thus, in the limit of very large squashing meron anisotropy has a well defined limit

On the other hand

$$\frac{c_{(3)}}{c_{(1)}} = \sqrt{\frac{2}{\alpha}} - \frac{5\sqrt{2}}{8}\sqrt{\alpha} + O(\alpha^{3/2})$$

as $\alpha\to 0$ So that in this limit merons become very anisotropic

On the other hand (α, ρ_0) are determined by the following algebraic equations:

$$\begin{cases} \Lambda = \dfrac{4-3\alpha^2}{4\rho_0^2} + \dfrac{2(c_{(2)})^2(c_{(3)}-1)^2 - \alpha^2\left((c_{(2)})^2 - c_{(3)}\right)}{4\kappa e^2\alpha^2\rho_0^4} \\ \Lambda = \dfrac{\left((c_{(2)})^2 - c_{(3)}\right)^2}{4\kappa e^2\rho_0^4} + \dfrac{\alpha^2}{4\rho_0^2} \end{cases}$$

The above algebraic equations fix the squashing parameter α as function of (κ, e, Λ)

However, it is easier to express Λ in terms of α and of the other parameter than the other way around

$$\Lambda = \begin{cases} \kappa e^2 + 4e^2\kappa\alpha + \frac{15}{2}\alpha^2 e^2\kappa + O(\alpha^3) & \text{as } \alpha\to 0^+ \\ 35\kappa e^2 + \frac{544}{5}\kappa e^2(\alpha-1) + O(\alpha-1)^2 & \text{as } \alpha-1\to 0 \\ \frac{27}{8}e^2\kappa\alpha^6 + \frac{117}{8}e^2\kappa\alpha^4 + 15e^2\kappa\alpha^2 + \frac{53}{27}e^2\kappa + O(\alpha^{-2}) & \text{as } \alpha\to\infty \end{cases}$$

Thus, one can note that the cosmological constant is positive when $0<\alpha<\infty$

More details in JHEP 11 (2023) 146

Gravitational Instantons in Einstein-Skyrme theory

As a last application of the present approach let us construct analytic examples of Instantons (Regular Euclidean Solutions) of Einstein Skyrme theory with non-vanishing Baryonic charge

Now it is clear $\rightarrow$ Metric and matter field Ansätze must be linked through the Vielbein

$$\begin{cases} ds^2 = \dfrac{dr^2}{f(r)} + (r^2 + r_0^2)\left(\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2}\right) \\ U = \exp(t_3\theta)\exp(t_2\psi)\exp(t_3\varphi) \end{cases}$$

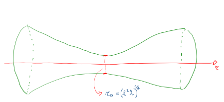
$\rightarrow$ the Skyrme field equations are identically satisfied and

$$\begin{cases} f(r) = \dfrac{r^2}{\ell^2} + 2\dfrac{\sqrt{\lambda}}{\ell}\,; \\ r_0 = (\ell^2\lambda)^{1/4}\,; \quad B = 1\,; \end{cases}$$

$\Uparrow$

thus, the wormhole throat exist due to the presence of Skyrme coupling λ

Note that unlike what happens with Instantons in EYM (where $T_{\mu\nu}^{YM} = 0$ due to self-duality) Instantons in Skyrme theory Back-react on the metric (more details in PRD 111, 084072 (2025))



Conclusions

*) I described the first analytic techniques to construct gravitating Skyrmions with $B \neq 0$

*) It is possible to adopt these techniques to find the first analytic Skyrmions on flat spaces identifying an integrable sector of Skyrme theory in (3+1)D with $B \neq 0$

*) These exact solutions represent Baryonic layers as well as ordered arrays of Baryonic tubes with interesting physical properties

*) these techniques can also be extended to EYM theory → we constructed the squashed 3-sphere as an exact solution of EYM theory sourced by anisotropic merons

*) Instantons with $B \neq 0$ can also be constructed in Einstein-Skyrme theory and one can explicitly see that the throath is supported by Skyrme coupling

Perspectives

*) $SU(2) \to SU(N)$

*) squashed solutions in Einstein-Skyrme

*) In the flat case → add Isospin Chemical potential

and many more!

Thank you very much!