Differential Curvature invariants as detectors of horizon and ergosurface radii for accelerating, rotating and charged black holes in (anti-)de Sitter background

G. V. Kraniotis

University of Ioannina

gvkraniotis@gmail.com
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- We determine novel SPIs and investigate their properties.
- The question of using differential curvature invariants for detecting stationary horizons of black holes is very important
- Indeed, some of the SPIs of the general black hole solutions we calculated, can be used as detectors of horizon and ergosurfaces radii.

The Bianchi identities and the Karlhede invariant

Using its covariant form, the classical Bianchi identities:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{1}$$

In eqn(1), the symbol ';'denotes covariant differentiation. Karlhede and collaborators introduced the following coordinate-invariant and Lorentz invariant object formed from the covariant derivative of the curvature tensor, the so called Karlhede invariant Karlhede et al, Gen.Rel.Grav 14 (1982),1159 :

$$\mathfrak{K} = R^{\lambda\mu\nu\kappa;\eta} R_{\lambda\mu\nu\kappa;\eta}. \tag{2}$$

For the Schwarzschild black hole the Karlhede invariant is:

$$\mathfrak{K} = -\frac{240\left(6m^3r^3 - 3m^2r^4\right)}{r^{12}}. \tag{3}$$

It vanishes and changes sign at the Schwarzschild event horizon $r = 2m$.
Unfortunately, this intriguing result does not generalises to the case of the Kerr black hole.

$$\mathfrak{K}^{Kerr} = \frac{720m^2}{\left(r^2 + a^2\cos(\theta)^2\right)^9}\left(\cos(\theta)^4 a^4 - 4\cos(\theta)^3 a^3 r - 6\cos(\theta)^2 a^2 r^2 + 4\cos(\theta)\, a\, r^3 + r^4\right)$$
$$\left(\cos(\theta)^4 a^4 + 4\cos(\theta)^3 a^3 r - 6\cos(\theta)^2 a^2 r^2 - 4\cos(\theta)\, a\, r^3 + r^4\right)\left(a^2\cos(\theta)^2 - 2mr + r^2\right). \tag{4}$$

In Boyer and Lindquist coordinates the event and Cauchy (inner) horizons of the Kerr black hole are located on the surface defined by $\Delta(r) \equiv r^2 + a^2 - 2mr = 0$, and are denoted by r_+ and r_- respectively.

Remark

However, we note that $\mathfrak{K}^{Kerr}$ it vanishes on the infinite-redshift surfaces, where $g_{tt} = 0$. Equivalently at the roots of the quadratic equation:

$$r^2 + a^2 \cos(\theta)^2 - 2mr = 0, \tag{5}$$

or

$$r_E^{\pm} = m \pm \sqrt{m^2 - a^2 \cos(\theta)^2}. \tag{6}$$

We also computed the SPI of first order:

$$\mathfrak{R} \equiv R_{\alpha\beta;\mu} R^{\alpha\beta;\mu}. \tag{7}$$

i.e. the scalar formed from the covariant derivative of the Ricci tensor for an accelerating Kerr-Newman-adS/dS black hole.

Abdelqader-Lake differential curvature invariants

Abdelqader and Lake Phys.Rev.D 91 (2015)693,084017, introduced the following curvature invariants and studied them for the Kerr metric:

$$I_1 \equiv C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \tag{8}$$

$$I_2 \equiv C^*_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \tag{9}$$

$$I_3 \equiv \nabla_\mu C_{\alpha\beta\gamma\delta} \nabla^\mu C^{\alpha\beta\gamma\delta}, \tag{10}$$

$$I_4 \equiv \nabla_\mu C_{\alpha\beta\gamma\delta} \nabla^\mu C^{*\alpha\beta\gamma\delta}, \tag{11}$$

$$I_5 \equiv k_\mu k^\mu, \tag{12}$$

$$I_6 \equiv l_\mu l^\mu, \quad I_7 \equiv k_\mu l^\mu, \tag{13}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $C^*_{\alpha\beta\gamma\delta}$ its dual, $k_\mu = -\nabla_\mu I_1$ and $l_\mu = -\nabla_\mu I_2$. They also defined the following invariants:

$$Q_1 \equiv \frac{1}{3\sqrt{3}} \frac{(I_1^2 - I_2^2)(I_5 - I_6) + 4 I_1 I_2 I_7}{(I_1^2 + I_2^2)^{9/4}}, \tag{14}$$

$$Q_2 \equiv \frac{1}{27} \frac{I_5 I_6 - I_7^2}{(I_1^2 + I_2^2)^{5/2}}, \tag{15}$$

$$Q_3 \equiv \frac{1}{6\sqrt{3}} \frac{I_5 + I_6}{(I_1^2 + I_2^2)^{5/4}}. \tag{16}$$

Page and Shoom Phys.Rev.Lett. 114 (2015),141102 noticed that, the invariant Q_2 was rewritten by Abdelqader and Lake as follows :

$$Q_2 = \frac{(I_6 + I_5)^2 - \left(\frac{12}{5}\right)^2 (I_1^2 + I_2^2)(I_3^2 + I_4^2)}{108(I_1^2 + I_2^2)^{5/2}}. \tag{17}$$

Their crucial observation was that the curvature invariant Q_2 can be expressed as the norm of the wedge product of two differential forms, namely:

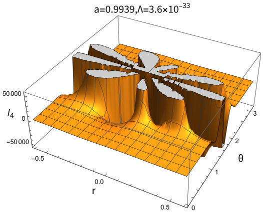
$$27(I_1^2 + I_2^2)^{5/2} Q_2 = 2 \left\| dI_1 \wedge dI_2 \right\|^2 , \tag{18}$$

where:

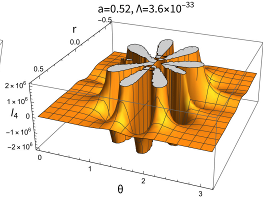
$$\left\| dI_1 \wedge dI_2 \right\|^2 = \frac{1}{2}((k_\mu k^\mu)(l_\nu l^\nu) - (k_\mu l^\mu)(k_\nu l^\nu)). \tag{19}$$

Under the light of this observation, the invariant Q_2 vanishes when the two gradient fields k_μ and l_μ are parallel. This led Page and Shoom to propose a generalisation of Q_2. For n polynomial curvature invariants $S^{(i)}$, they introduced an $n-$form differential invariant $W = dS^{(1)} \wedge \ldots \wedge S^{(n)}$, that vanishes on Killing horizons (i.e. $\| W \|^2 \overset{\circ}{=} 0$), in stationary spacetimes of local cohomogeneity n . For $n = 2$ we compute the invariant W for an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime and applying the Newman-Penrose formalism.

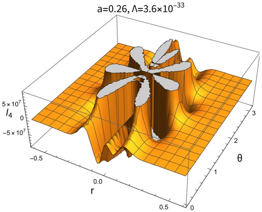
Analytic computation of SPIs for an accelerating KN(a)dS black hole



(a) 3D plot of I_4.



(b) 3D plot of I_4



(c) 3D Plot of I_4.

Figure: 3D plots of the differential curvature invariant, I_4, plotted as a function of the Boyer-Lindquist coordinates r and θ . (a) for high spin parameter $a = 0.9939$, charge $q = 0$, $\Lambda = 3.6 \times 10^{-33}$,$m = 1$. (b) For spin $a = 0.52$, charge $q = 0$,$\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (c) For low spin $a = 0.26$, electric charge $q = 0$,$\Lambda = 3.6 \times 10^{-33}$ and mass $m = 1$.

Theorem

We computed the following explicit analytic expression for the differential invariant I_5 for the spacetime of the Kerr-(anti-)de Sitter black hole:

$$
\begin{aligned}
I_5 = &-\frac{27648m^4}{\left(r^2+a^2\cos(\theta)^2\right)^{15}}\Bigg(\Lambda\cos(\theta)^{18}a^{18} - a^{16}\left(\Lambda a^2+42\Lambda r^2-3\right)\cos(\theta)^{16} + 42a^{14}\Bigg(\left(\Lambda r^2-\frac{1}{14}\right)a^2 \\
&+\frac{73\Lambda r^4}{6}-3r^2\Bigg)\cos(\theta)^{14} - 462\left(\left(\Lambda r^2+\frac{1}{22}\right)a^2 - \frac{7\left(-\frac{205}{42}\Lambda r^3+m+\frac{33}{7}r\right)r}{11}\right)r^2a^{12}\cos(\theta)^{12} \\
&+994\left(\left(\Lambda r^2-\frac{9}{142}\right)a^2-\frac{210\left(-\frac{7}{20}\Lambda r^3+m+\frac{71}{70}r\right)r}{71}\right)r^4a^{10}\cos(\theta)^{10} \\
&+\left(-105a^{10}r^6+\left[1029\Lambda r^{10}+9114m\,r^7\right]a^8\right)\cos(\theta)^8 - 994\Bigg[\left(\Lambda r^2+\frac{15}{142}\right)a^2+\frac{205\Lambda r^4}{142}+\frac{636mr}{71} \\
&-3r^2\Bigg]r^8a^6\cos(\theta)^6 + 462r^{10}a^4\left(\left(\Lambda r^2-\frac{3}{22}\right)a^2+\frac{73r\left(\frac{1}{6}\Lambda r^3+m-\frac{33}{73}r\right)}{11}\right)\cos(\theta)^4 \\
&-42r^{12}\left(\left(\Lambda r^2+\frac{1}{2}\right)a^2+\Lambda r^4+6mr-3r^2\right)a^2\cos(\theta)^2 + r^{14}\left(\left(\Lambda r^2-3\right)a^2+\Lambda r^4+6mr-3r^2\right)\Bigg).
\end{aligned}
\tag{20}
$$

Corollary

For $\theta = \pi/2$, i.e. for the equatorial plane, I_5 in eqn. (20) reduces to:

$$I_5 = -\frac{27648\left(\left(\Lambda r^2 - 3\right)a^2 + \Lambda r^4 + 6mr - 3r^2\right)m^4}{r^{16}}. \tag{21}$$

Thus I_5 in this case, vanishes at the stationary horizons of the Kerr black hole with cosmological constant.

Corollary

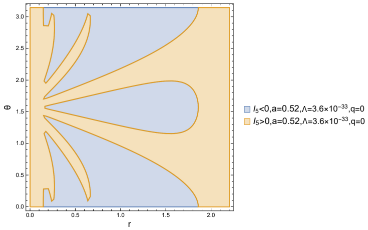
For $\theta = 0$, i.e. along the axis, I_5 reduces to:

$$I_5 = -\frac{1354752m^4\left(a^6 - 5a^4r^2 + 3a^2r^4 - \frac{1}{7}r^6\right)^2 r^2\left(\Lambda r^4 + \left(\Lambda a^2 - 3\right)r^2 + 6mr - 3a^2\right)}{\left(a^2 + r^2\right)^{15}}. \tag{22}$$

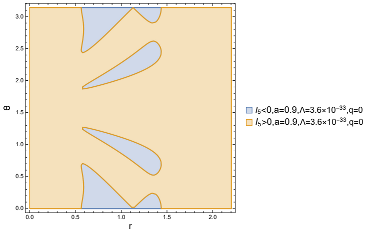
For all the values of the Kerr-parameter a consistent with a Kerr-(anti-)de Sitter black hole, the norm of the gradient vector k_μ (i.e. I_5) vanishes at the stationary black hole horizons and at the real positive roots of the sextic radial polynomial:

$$r \simeq 0.481574618807a,\ 1.2539603376a,\ 4.3812862675a. \tag{23}$$

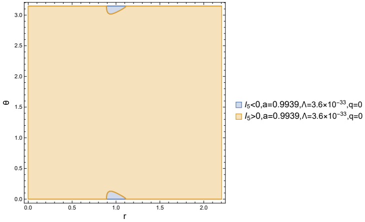
Whereas the vanishing of I_5 in the equatorial plane (and on the axis away from the discrete roots in Eqn.(23)) singles out the horizons the global behaviour of I_5 is also interesting. In general terms, the area of regions where $I_5 < 0$ (this means that in these regions the vector k_μ is timelike) on any $r - \theta$ hypersurface decreases as the Kerr parameter increases. This strong sign dependence of the curvature invariant I_5 with black hole's spin a is amply demonstrated in the following Figures:



(a) Region plot of gradient curvature invariant I_5.



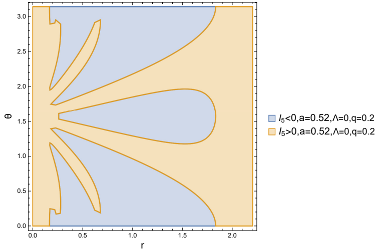
(b) Region plot of gradient curvature invariant I_5.



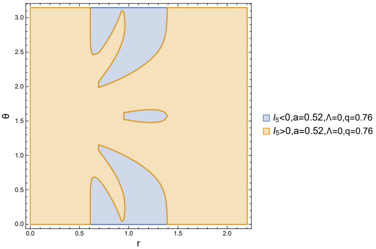
(c) Region Plot of gradient curvature invariant I_5.

Figure: Region plots of the curvature invariant I_5 , Eqn.(20), for the Kerr-(anti-)de Sitter black hole. (a) for spin parameter $a = 0.52$, charge $q = 0$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (b) For spin $a = 0.9$, charge $q = 0$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (c) For high spin $a = 0.9939$, electric charge $q = 0$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$ and mass $m = 1$. We note that the area of regions of negative sign of the invariant I_5 (and hence the regions where the gradient vector $k_\mu = -\nabla_\mu (C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta})$ is timelike) decreases as the Kerr parameter a increases.

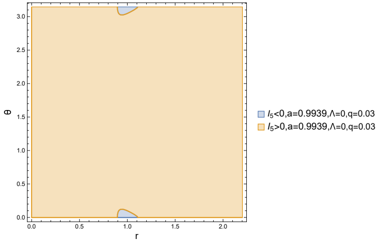
We reiterated the analysis of the I_5 invariant for the case of the Kerr-Newman black hole (KN BH) in Figure 3. The strong sign dependence on the black hole's spin is still present. However, there is now a new effect due to the electric charge. For a fixed value of the Kerr parameter a, increasing the electric charge results in a decrease of the area of regions where $I_5 < 0$ and thus the areas in the $r - \theta$ space where the gradient vector k_μ is timelike. As a matter of fact for the choice of values $a = 0.9939, q = 0.11$ the gradient vector k_μ is spacelike throughout the $r - \theta$ space, as shown in the following Figures:



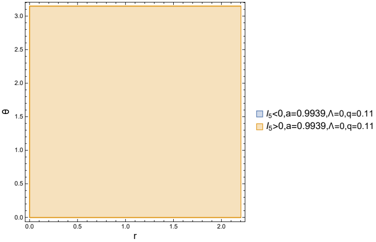
(a) Region plot of curvature invariant I_5 for KN BH.



(b) Region plot of curvature invariant I_5 for KN BH.



(c) Region Plot of curvature invariant I_5 for KN BH.



(d) Region Plot of the invariant I_5 for KN BH.

Figure: Region plots of the curvature invariant I_5, for the Kerr-Newman black hole. (a) for spin parameter $a = 0.52$, charge $q = 0.2$, $m = 1$. (b) For spin $a = 0.52$, charge $q = 0.76$, $m = 1$. (c) For high spin $a = 0.9939$, electric charge $q = 0.03$, and mass $m = 1$. (d) For high spin $a = 0.9939$, electric charge $q = 0.11$, and mass $m = 1$.

As regards the values of I_5 in the equatorial plane and on the axis for a KN(a-)dS black hole we obtain:

Corollary

$$I_5(\theta=\frac{\pi}{2}) = -\frac{27648\left(\Lambda r^4+\left(\Lambda a^2-3\right)r^2+6mr-3a^2-3q^2\right)\left(mr-q^2\right)^2\left(mr-\frac{4q^2}{3}\right)^2}{r^{20}}, \qquad (24)$$

and

Corollary

$$I_5(\theta=0) = -\frac{3072\left(\Lambda a^2r^2+\Lambda r^4-3a^2+6mr-3q^2-3r^2\right)}{\left(a^2+r^2\right)^{15}}\Big(21a^6m^2r-5a^6m\,q^2-105a^4m^2r^3$$
$$+85a^4m\,q^2r^2-12a^4q^4r+63a^2m^2r^5-95a^2m\,q^2r^4+32a^2q^4r^3-3r^7m^2+7m\,q^2r^6-4q^4r^5\Big)^2. \qquad (25)$$

Theorem

The exact analytic expression for the differential curvature invariant Q_1 for the Kerr-(anti-) de Sitter black hole is the following:

$$Q_1 = -\frac{\left(\Lambda \cos(\theta)^4 a^4 - \Lambda \cos(\theta)^2 a^4 - \Lambda a^2 r^2 - \Lambda r^4 + 3a^2 \cos(\theta)^2 - 6mr + 3r^2\right)\left(a^2 \cos(\theta)^2 - r^2\right)}{3\left(r^2 + a^2 \cos(\theta)^2\right)^{\frac{3}{2}} m}. \tag{26}$$

Corollary

The invariant Q_1 vanishes on the boundary of the ergosphere region and at $r = \pm a\cos(\theta)$. Q_1 is strictly positive outside the outer ergosurface, vanishes at the ergosurface, and then becomes negative as soon as we cross it. The surfaces associated with the roots of Q_1 at the inner ergosurface and at $r = \pm a\cos(\theta)$, for the observed Λ lie strictly within the outer ergosurface regardless of the values of m and a. Thus, these additional roots of Q_1 do not affect its capability to detect the outer ergosurface. We conclude that Q_1 is a very convenient invariant to use for detecting the ergosurface in Kerr-de Sitter spacetime.

The Plebański-Demiański Ann.Phys.98 (1976)98 metric covers a large class of Einstein-Maxwell electrovacuum (algebraic of Petrov type D) solutions which include the physically most significant case: that of an accelerating, rotating and charged black hole with a non-zero cosmological constant. We focus on the following metric that describes an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime Podolsky Griffiths, Phys.Rev.D73 (2006)044018 :

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\Bigg\{-\frac{Q}{\rho^2}\left[\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2\Bigg\}, \tag{27}$$

where (α is the acceleration of the black hole):

$$\Omega = 1-\alpha r\cos\theta,\ \rho^2 = r^2 + a^2\cos^2\theta, \tag{28}$$

$$P = 1-2\alpha m\cos\theta + \left(\alpha^2(a^2+q^2) + \frac{1}{3}\Lambda a^2\right)\cos^2\theta, \tag{29}$$

$$Q = ((a^2+q^2) - 2mr + r^2)(1-\alpha^2 r^2) - \frac{1}{3}\Lambda(a^2+r^2)r^2. \tag{30}$$

A subcase of the metric (27) for $a = 0$, is the charged C-metric with a cosmological constant. For $\alpha = 0$, the metric (27) reduces to that for the Kerr-Newman-(anti-)de Sitter spacetime. The metric (27) becomes singular at the roots of Ω, ρ^2, Q, P. Some of them are pseudosingularities (mere coordinate singularities) while others are true (curvature) singularities detected by the curvature invariants.

We add the following remarks. The expression for Q is a quartic and, for the physically significant solutions, it must generally (for $\alpha^2 + \frac{1}{3}\Lambda > 0$) have up to three positive real roots. Such roots would correspond to inner (Cauchy) and outer (event) black hole horizons and to an acceleration horizon.

On the other hand, the case in which $\alpha^2 + \frac{1}{3}\Lambda \leq 0$ is of a very distinct character. This condition implies that there is no acceleration horizon, so that the space-time represents at most a single black hole with inner (Cauchy) and outer black hole (event) horizons. Furthermore, such a source is accelerating in a background with negative Λ. In this case, Q factorises as follows:

$$Q = -(\alpha^2 + \frac{1}{3}\Lambda)(r - r_0)(r - r_1)(r - r_-)(r - r_+), \tag{31}$$

where the event horizon corresponds to the largest positive real root r_+ while the Cauchy horizon is denoted by r_- and obeys $0 < r_- < r_+$. The latter configuration has been referred as the slowly accelerating Kerr-Newman-adS black hole Podolsky Griffiths, CUP (2009), Anabalon et al JHEP (2019)96.

Theorem

We calculated the exact algebraic expression for the invariant Q_2 for the accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime. Our result is:

$$Q_2 = \frac{-1}{\mathcal{A}}\Bigg(\alpha\left(\left(\left(a^4m^2r + 2a^2m\,q^2r^2 + \frac{8}{9}q^4r^3\right)\alpha^2 + a^2m\left(mr - \frac{2q^2}{3}\right)\right)\cos(\theta)^3\right.$$
$$+\left(\left(a^4m^2 + 2a^2m\,q^2r - 2m\,q^2r^3 + \frac{16}{9}q^4r^2\right)\alpha^2 + a^2m^2\right)\cos(\theta)^2 + \alpha\Bigg(a^2\alpha^2m^2r^3 + 2a^2m\,q^2 + r^3m^2$$
$$-2m\,q^2r^2 + \frac{16}{9}q^4r\Bigg)\cos(\theta) + m\,r^2\left(a^2m + \frac{2q^2r}{3}\right)\alpha^2 + r^2m^2 - 2m\,q^2r + \frac{8q^4}{9}\Bigg)^2\Bigg[r^2\left(a^2 - 2mr + q^2 + r^2\right)\alpha^2$$
$$+\frac{\Lambda r^4}{3} + \left(\frac{\Lambda a^2}{3} - 1\right)r^2 + 2mr - a^2 - q^2\Bigg]a^2\sin(\theta)^2\left(1 - 2\alpha m\cos(\theta) + \left(\alpha^2\left(a^2+q^2\right) + \frac{\Lambda a^2}{3}\right)\cos(\theta)^2\right). \tag{32}$$

where

$$\mathcal{A} \equiv (\alpha r\cos(\theta) - 1)^4\left(\left(\left(a^2m + q^2r\right)^2\alpha^2 + a^2m^2\right)\cos(\theta)^2 + 2q^2\alpha\left(a^2m - m\,r^2 + q^2r\right)\cos(\theta)\right.$$
$$\left. + r^2m^2\alpha^2a^2 + \left(mr - q^2\right)^2\right)^3. \tag{33}$$

Corollary

We observe that Eqn.(32) vanishes at the radii of the horizons of the accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime. Thus, the differential curvature invariant Q_2 can serve as horizon detector for the most general class of accelerating rotating and charged black holes with non-zero cosmological constant.

Corollary

In the equatorial plane $\theta = \frac{\pi}{2}$, Q_2 *reduces to:*

$$Q_2(\theta = \frac{\pi}{2}) = -\frac{\left(m\,r^2\left(a^2 m + \frac{2q^2 r}{3}\right)\alpha^2 + r^2 m^2 - 2m\,q^2 r + \frac{8q^4}{9}\right)^2}{\left(r^2 m^2 \alpha^2 a^2 + (mr - q^2)^2\right)^3}$$
$$\times \left(r^2\left(a^2 - 2mr + q^2 + r^2\right)\alpha^2 + \frac{\Lambda r^4}{3} + \left(\frac{\Lambda a^2}{3} - 1\right) r^2 \right.$$
$$\left. + 2mr - a^2 - q^2\right) a^2. \tag{34}$$

The vanishing of Q_2 *in the equatorial plane singles out the horizons (away from the discrete roots of the cubic radial polynomial in (34)).*

On the axis $Q_2 = 0$.

Examples

For the choice of parameters:
$a = 0.9939, q = 0.11, \alpha = 0.05, m = 1, \Lambda = 3.6 \times 10^{-33}$ the event and Cauchy horizon radii are located at: $r_+ = 1.00792, r_- = 0.992076$ respectively, whereas the acceleration horizon is located at $r_\alpha = 20$. The discrete roots of the cubic in (34) are:
$r \simeq 0.0160536, 0.00808674, -49709.3$. For the parameters:
$a = 0.52, \alpha = 0.1, q = 0.3, m = 1, \Lambda = 3.6 \times 10^{-33}$, we obtain $r_+ = 1.79975, r_- = 0.20025, r_\alpha = 10$ for the event, Cauchy and acceleration horizon radii respectively. The discrete roots of the cubic in this case are $r \simeq 0.119334, 0.0601658, -1671.35$. For a black hole with: $a = 0.52, q = 0.005, m = 1, \Lambda = -3, \alpha = 0.9$, the horizons radii are: $r_+ = 0.706123, r_- = 0.149774$, whereas the cubic has one negative real root and two non-real complex conjugate roots. Likewise, for a slowly accelerating KNadS black hole, for the set of values of the parameters: $a = 0.52, q = 0.005, \alpha = 0.0014, \Lambda = -0.006, m = 1$ the event and Cauchy horizons are located at $r_+ = 1.83955, r_- = 0.145856$ respectively, whereas the extra discrete roots of the cubic are located at
$r \simeq 0.000033333, 0.0000166667, -3.06123 \times 10^{10}$.

Remark

Thus we conclude, that the extra discrete roots of the invariant Q_2 do not affect its capability to detect the black hole's event and acceleration horizon radii. We also note that our conclusions for the invariant Q_2 are valid off of the equatorial plane. Indeed, we have repeated the above calculations for different angles i.e. for $\theta \neq \pi/2$ and still the invariant Q_2 detects the horizons radii whereas the extra discrete roots do not affect its detection capabilities.

Bianchi identities in terms of the spin coefficients and the Weyl and Ricci scalars read as follows Chandrasekhar:

$$-D\Psi_2 + 3\varrho\Psi_2 + 2\varrho\Phi_{11} - 2D\Lambda = 0, \quad (35)$$

$$-\Delta\Psi_2 - 3\mu\Psi_2 - 2\mu\Phi_{11} - 2\Delta\Lambda = 0, \quad (36)$$

$$-\delta^*\Psi_2 - 3\pi\Psi_2 + 2\pi\Phi_{11} - 2\delta^*\Lambda = 0, \quad (37)$$

$$-\delta\Psi_2 + 3\tau\Psi_2 - 2\tau\Phi_{11} - 2\delta\Lambda = 0. \quad (38)$$

The directional derivatives are given by the expressions: $D = l^\mu\partial_\mu, \Delta = n^\mu\partial_\mu, \delta = m^\mu\partial_\mu, \delta^* = \overline{m}^\mu\partial_\mu$.

In order to calculate the Ricci rotation coefficients that appear in Bianchi identities, Eqns.(35)-(38), we will work with the following null-tetrad:

$$l^\mu = \left(-\frac{\sqrt{2}}{2}\frac{(r^2+a^2)\Omega}{\rho\sqrt{Q}}, \frac{-\Omega\sqrt{2}\sqrt{Q}}{2\rho}, 0, -\frac{\Omega\sqrt{2}a}{2\sqrt{Q}\rho}\right), \quad (39)$$

$$n^\mu = \left(-\frac{\Omega\sqrt{2}(a^2+r^2)}{2\rho\sqrt{Q}}, \frac{\Omega\sqrt{Q}\sqrt{2}}{2\rho}, 0, -\frac{\Omega\sqrt{2}a}{2\sqrt{Q}\rho}\right), \quad (40)$$

$$m^\mu = \left(-\frac{\Omega a\sin(\theta)\sqrt{2}}{2\rho\sqrt{P}}, 0, -\frac{i\sqrt{2}\Omega\sqrt{P}}{2\rho}, -\frac{\Omega\sqrt{2}}{2\sqrt{P}\rho\sin(\theta)}\right), \quad (41)$$

$$\bar{m}^\mu = \left(-\frac{\Omega\sqrt{2}a\sin(\theta)}{2\rho\sqrt{P}}, 0, \frac{i\Omega\sqrt{2}\sqrt{P}}{2\rho}, -\frac{\Omega\sqrt{2}}{\sqrt{P}2\rho\sin(\theta)}\right). \quad (42)$$

We computed the Ricci-rotation coefficients via the formula for the λ-symbols given by:

$$\lambda_{(a)(b)(c)} = e_{(b)i,j}[e_{(a)}^{i} e_{(c)}^{j} - e_{(a)}^{j} e_{(c)}^{i}]. \tag{43}$$

This formula has the advantage that one has to calculate ordinary derivatives of the dual co-tetrad. The Ricci rotation coefficients $\gamma_{(a)(b)(c)}$ are expressed through the λ-coefficients as follows:

$$\gamma_{(a)(b)(c)} = \frac{1}{2}[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}] \tag{44}$$

Using the null-tetrad in Eqns.(39)-(42), for the accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime, we computed the following new mathematical formulae for the Ricci spin coefficients that appear in the Bianchi identities:

$$\mu = \varrho = -\frac{\sqrt{Q}\alpha\cos(\theta)}{\sqrt{2}\rho} - \frac{\Omega\sqrt{Q}(r - ia\cos(\theta))}{\sqrt{2}\rho^3}, \tag{45}$$

$$\tau = \pi = \frac{i\sqrt{P}}{\sqrt{2}\rho}\alpha r\sin(\theta) - \frac{\sin(\theta)\sqrt{P}\Omega a}{\sqrt{2}\rho^3}(r - ia\cos(\theta)). \tag{46}$$

The only non-zero curvature scalars in the NP-formalism for the metric (27) using the null-tetrad in Eqns.(39)-(42), are the Weyl scalar Ψ_2 which is given by the following closed form expression:

$$\begin{aligned}\Psi_2 &\equiv C_{\mu\nu\lambda\sigma}\overline{m}^{\mu}n^{\nu}l^{\lambda}m^{\sigma} \\ &= -\frac{(\alpha r\cos(\theta)-1)^3\left(\left(\alpha r q^2+(a\alpha+i)\,ma\right)\cos(\theta)+m\,(ia\alpha-1)\,r+q^2\right)}{-\cos(\theta)^4a^4+2\,ia^3r\cos(\theta)^3+2\,ir^3a\cos(\theta)+r^4} \\ &= \frac{\left(-m\,(1-ia\alpha)+\frac{q^2(1+\alpha r\cos(\theta))}{r-ia\cos(\theta)}\right)(1-\alpha r\cos(\theta))^3}{(r+ia\cos(\theta))^3},\end{aligned} \tag{47}$$

and the Ricci-NP scalars:

$$\Phi_{11}\equiv\frac{1}{4}R_{\mu\nu}(l^{\mu}n^{\nu}+m^{\mu}\overline{m}^{\nu})=\frac{(\alpha r\cos(\theta)-1)^4q^2}{2\left(r^2+a^2\cos(\theta)^2\right)^2},\quad\text{and}\quad\Lambda. \tag{48}$$

Now we have at our disposal all the arsenal necessary to prove the following theorem of the Page-Shoom invariant for the case of accelerating Kerr-Newman black holes in (anti-)de Sitter spacetime:

Theorem

We computed in closed-form the Page-Shoom invariant for accelerating, rotating and charged black holes with non-zero cosmological constant ($\Lambda \neq 0$) in the Newman-Penrose formalism with the result:

$$W \equiv \left\| \mathrm{d}\Psi_2 \wedge \mathrm{d}\overline{\Psi}_2 \right\|^2 = 16\Re[(3\mu\Psi_2 + 2\mu\Phi_{11})^2(-3\bar{\pi}\overline{\Psi}_2 + 2\bar{\pi}\Phi_{11})^2] - 16\|3\mu\Psi_2 + 2\mu\Phi_{11}\|^2\|-3\pi\Psi_2 + 2\pi\Phi_{11}\|^2. \tag{49}$$

In (49), the Weyl scalar Ψ_2 and the Ricci-NP scalar Φ_{11} are given by closed-form expressions,equations (47) and (48), respectively. Whereas the spin coefficients μ, π are given by the explicit algebraic expressions in eqns.(45),(46).

Proof.

Using the expression for the covariant derivative,

$$\nabla_\mu = n_\mu D + l_\mu \Delta - \bar{m}_\mu \delta - m_\mu \delta^*. \tag{50}$$

the Bianchi identities, eqns.(35)-(38) and our computation for the spin coefficients, eqns.(45)-(46) we obtain eqn.(49). □

Corollary

From Eqn.(45), it is evident that $\mu = \varrho$ vanishes at the stationary horizons. This follows from the fact that the real roots of the radial polynomial Q (eqn. 30), yield coordinate singularities which correspond to the up to four horizons of the spacetime. As a result the invariant W must vanish there as well.

Cartan Invariants as detectors of the horizon radii for accelerating KN(a)dS BH

The Ricci spin coefficients μ, ϱ, τ, π in Eqns.(45) and (46) are Cartan invariants for these accelerating black hole solutions in (anti-)de Sitter spacetime

Important theorem

Thus the Cartan invariants:

$$\mu = \varrho = -\frac{\sqrt{Q}\alpha\cos(\theta)}{\sqrt{2}\rho} - \frac{\Omega\sqrt{Q}(r - ia\cos(\theta))}{\sqrt{2}\rho^3} \tag{51}$$

can serve as detectors of the horizon radii for the important class of accelerating Kerr-Newman black holes in (anti-)de Sitter spacetime.

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- Our novel results on the differential invariants for such general black hole solutions indicate that the teleological nature of event horizon has been tamed.