The Dynamics of Scalar-Field Quintom Cosmological Models

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J. Tot[1] B. Yildirim[1] A. Coley[1] **Genly Leon** [2,3,4]

[1]Department of Mathematics and Statistics, Dalhousie University

[2]Departamento de Matemáticas, Universidad Católica del Norte

[3]Institute of Systems Science, Durban University of Technology

[4]Departamento de Física, Universidad de Antofagasta, Antofagasta 1240000, Chile

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http://www.hsrgc.gr info@hsrgc.gr

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Institutional Context and Funding

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Authors and Affiliations

- J. Tot, B. Yildirim, A. Coley — Dalhousie University
- Genly Leon — Universidad Católica del Norte (associated professor), Universidad de Antofagasta (Phd student in Phys.), Durban University of Technology (associated Researcher)

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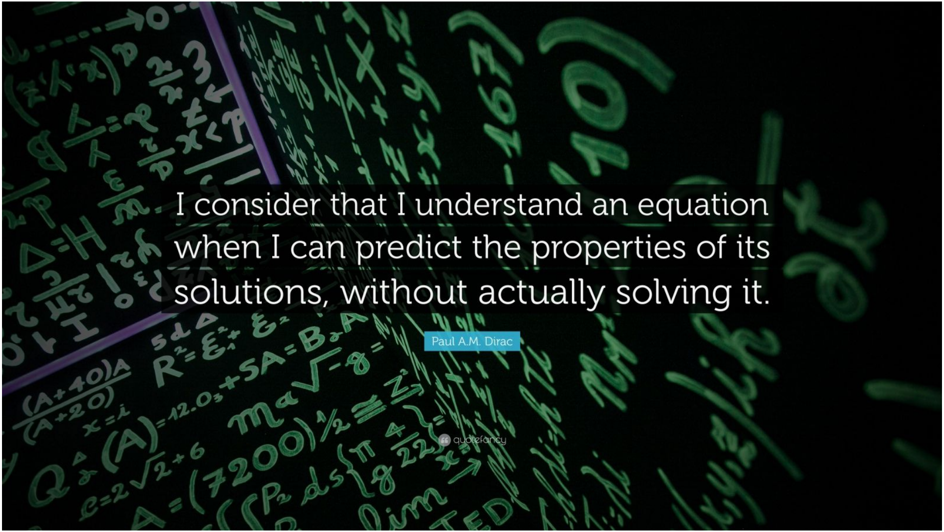
Outline



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Dynamical Systems and Gravity Models in Cosmology

Phase Space and Evolution

- Cosmological models $\mathcal{M}$ evolve as states $\mathbf{x} \in \Sigma$
- Dynamics governed by constrained PDEs:

$$\partial_t \mathbf{x} = \mathbf{X}(\mathbf{x}, \partial_i \mathbf{x}, \ldots), \quad \mathbf{C}(\mathbf{x}, \partial_i \mathbf{x}, \ldots) = 0$$

- For homogeneous models:

$$\frac{dy}{d\tau} = f(y), \quad y \in \mathbb{R}^n, \quad g(y) = 0$$

Analysis Procedure

1. Compactify state space
2. Identify invariant sets and symmetries
3. Locate singular points and analyze stability
4. Construct Dulac or monotonic functions
5. Study bifurcations and transitions
6. Deduce asymptotic behavior
7. Build heteroclinic sequences

Gravity Models

General Relativity (GR):

Model	$\mathcal{L}_\phi$	Equation of Motion
Quintessence	$-V(\phi) + X$	$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$
Tachyon	$-V(\phi)\sqrt{1-2X}$	$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V}\frac{dV}{d\phi} = 0$
Phantom	$-V(\phi) - X$	$\ddot{\phi} + 3H\dot{\phi} - \frac{dV}{d\phi} = 0$
K-essence	$L(\phi, X)$	$\left(\frac{\partial L}{\partial X} + 2X\frac{\partial^2 L}{\partial X^2}\right)\ddot{\phi} + \frac{\partial L}{\partial X}(3H\dot{\phi}) + \frac{\partial^2 L}{\partial\phi\partial X}\dot{\phi}^2 - \frac{\partial L}{\partial \phi} = 0$

$X \equiv -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$

Modified Gravity (MG):

- Scalar-tensor theories
- $f(R)$, $f(R_{ab}R^{ab})$, higher-order curvature models

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Quintom Cosmology: Dynamics, Curvature, and Perturbations

Scope: Dynamical analysis of scalar-field Quintom models with exponential potentials and dual-field structure:

- Quintessence ($w > -1$)
- Phantom ($w < -1$)

Key Features:

- Transitions across the phantom divide ($w = -1$)
- Multiple inflationary epochs and non-singular bouncing solutions
- Compact phase-space formulation with critical point classification
- Physical interpretation of attractors and trajectories

Spatial Curvature:

- Explicitly incorporated into the dynamical system
- Inflationary solutions remain stable under curvature effects

Perturbation Framework:

- Linear perturbations in Newtonian gauge
- Gauge-invariant variables in extended phase space
- Enhanced predictive power for structure formation and cosmic history

Cosmological Model: Action and Dynamics

Action:

$$S = \int \sqrt{-g}\, d^4x \left[R - \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} e^{\kappa\phi} (\nabla\psi)^2 - V_0 e^{\lambda\phi} \right]$$

- ϕ: Quintessence field, ψ: Phantom field
- κ: Coupling constant, $V(\phi) = V_0 e^{\lambda\phi}$

Metric (FLRW, flat case $K = 0$):

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \right]$$

Field Equations:

$$3H^2 = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2} e^{\kappa\phi} \dot{\psi}^2 + V(\phi)$$

$$\ddot{\phi} + 3H\dot{\phi} + \tfrac{\kappa}{2} e^{\kappa\phi} \dot{\psi}^2 + V'(\phi) = 0$$

$$\ddot{\psi} + 3H\dot{\psi} + \kappa\dot{\phi}\dot{\psi} = 0$$

- $H = \dot{a}/a, \quad V'(\phi) = \lambda V_0 e^{\lambda\phi}$

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Dimensionless Variables and Autonomous System

Normalized variables:

$$\chi^2 = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \quad V(\phi) = V_0 e^{\lambda\phi}$$

$$h^2 = \tfrac{3H^2}{\chi^2}, \quad \eta^2 = \tfrac{e^{\kappa\phi}\dot{\psi}^2}{2\chi^2}, \quad \Phi^2 = \tfrac{\dot{\phi}^2}{2\chi^2}, \quad \Psi = \tfrac{V(\phi)}{\chi^2}$$

$$\text{Constraint:} \quad h^2 + \eta^2 = \Phi^2 + \Psi = 1$$

Bounded phase space:

$$(\Phi, h) \in [-1, 1] \times [-1, 1]$$

Time reparametrization:

$$d\tau = \chi\, dt, \quad f' = \frac{df}{d\tau}$$

Autonomous system:

$$\Phi' = -\tfrac{(1-\Phi^2)}{\sqrt{2}} \left(\kappa(1-h^2) + \sqrt{6}h\Phi + \lambda\right) \tag{1}$$

$$h' = \tfrac{(1-h^2)}{\sqrt{2}} \left(\kappa h\Phi + \sqrt{6}(1-\Phi^2)\right) \tag{2}$$

Additional relations:

$$\eta^2 = 1 - h^2, \quad \Psi = 1 - \Phi^2, \quad \frac{\dot{\chi}}{\chi^2} = -\frac{\Phi}{\sqrt{2}} \left(\kappa(1-h^2) + \sqrt{6}h\Phi\right)$$

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Equilibrium Points: A–D and $O_\pm$

Label	Existence	Sink	Saddle	Source	Inflation
A	$\forall\lambda, \kappa$	$\kappa > 0, \lambda < -1$	$\kappa > 0, \lambda > -1$, or $\kappa < 0, \lambda < -1$	$\kappa < 0, \lambda > -1$	N/A
B	$\forall\lambda, \kappa$	$\kappa < 0, \lambda > 1$	$\kappa < 0, \lambda < 1$, or $\kappa > 0, \lambda > 1$	$\kappa > 0, \lambda < 1$	N/A
C	$\forall\lambda, \kappa$	$\kappa > 0, \lambda < 1$	Same as B	$\kappa < 0, \lambda > 1$	N/A
D	$\forall\lambda, \kappa$	$\kappa < 0, \lambda > -1$	Same as A	$\kappa > 0, \lambda < -1$	N/A
O_+	$\forall\lambda, \kappa$	$\kappa < 0, \lambda < -\kappa$	$\kappa > 0, \lambda < -\kappa$ or $\kappa < 0, \lambda > -\kappa$	$\kappa > 0, \lambda > -\kappa$	N/A
O_-	$\forall\lambda, \kappa$	$\kappa > 0, \lambda > -\kappa$	Same as O_+	$\kappa < 0, \lambda < -\kappa$	N/A

Table: Equilibrium points A-D, $O_\pm$: existence and stability conditions.

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Inflationary Equilibrium Points

Label	Existence	Sink	Saddle	Source	Inflation
E	$\lvert\lambda\rvert < 1$	$\lvert\lambda\rvert < 1,$ $\kappa\lambda + \lambda^2 - 1 < 0$	$\lvert\lambda\rvert < 1,$ $\kappa\lambda + \lambda^2 - 1 > 0$	DNE	$\lvert\lambda\rvert < 1/\sqrt{3}$
F	Same as E	DNE	Same as E	$\lvert\lambda\rvert < 1,$ $\kappa\lambda + \lambda^2 - 1 < 0$	Same as E
$G,\ H$	$0 < \kappa(\kappa + \lambda)$ $\kappa\lambda + \lambda^2 - 1 < 0$	DNE	If exists	DNE	$\frac{2}{3} < \frac{\kappa}{\kappa+\lambda}$

Table: Equilibrium points E and F, G and H: existence and inflation conditions.

Local Stability via Eigenvalues (Simpsons Edition)

Sample eigenvalues with character tags:

A (Homer) : $(-\sqrt{2}\kappa, \sqrt{2}\lambda + 2\sqrt{3})$

B (Marge) : $(\sqrt{2}\kappa, 2\sqrt{3} - \sqrt{2}\lambda)$

C (Bart) : $(-\sqrt{2}\kappa, \sqrt{2}\lambda - 2\sqrt{3})$

D (Lisa) : $(\sqrt{2}\kappa, -\sqrt{2}\lambda - 2\sqrt{3})$

E (Mr. Burns) : $\left(\frac{\lambda^2-6}{2\sqrt{3}}, \frac{\lambda(\kappa+\lambda)-6}{\sqrt{3}}\right)$

F (Ned Flanders) : $\left(-\frac{\lambda^2-6}{2\sqrt{3}}, \frac{6-\lambda(\kappa+\lambda)}{\sqrt{3}}\right)$

Interior points:

$$\textbf{G (Professor Frink)} : \delta_\pm = \frac{\sqrt{3}\kappa \pm \sqrt{\Delta}}{2\sqrt{\kappa(\kappa+\lambda)+6}},$$

$$\textbf{H (Sideshow Bob)} : \Delta_\pm = \frac{-\sqrt{3}\kappa \pm \sqrt{\Delta}}{2\sqrt{\kappa(\kappa+\lambda)+6}}$$

Legend:

- Homer: source
- Marge: stabilizing influence
- Bart: sharp transitions
- Lisa: opposite to Lisa
- Mr. Burns: slow-roll inflation
- Flanders: graceful exit
- Frink: saddle with complicated dynamics
- Sideshow Bob: long inflation, elegant instability

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Note: Character names are mnemonic only—they do not literally represent eigenvalue behavior (though Moe might resemble a stable sink, and Comic Book Guy a saddle point with critical attitude).

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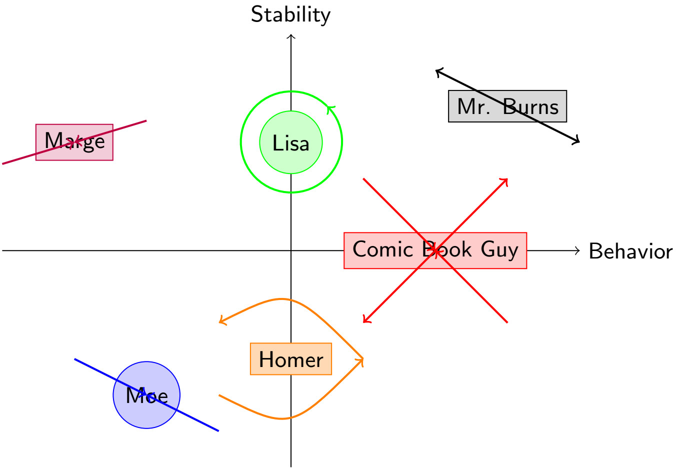
Springfield as a Dynamical System



Characters mapped to equilibrium types: Moe (sink), Comic Book Guy (saddle), Lisa (center), Homer (chaotic), Marge (stable node), Mr. Burns (source).

Inflationary Dynamics

Inflation condition:

$$q = -\frac{\dot{H}}{H^2} - 1 < 0 \quad \Rightarrow \quad q = \frac{3}{h^2}(h^2 + \Phi^2 - 1) - 1 < 0$$

Inflationary fixed points:

- E: boundary sink — single-field inflation.
- H: interior saddle — dual-field inflation.
- G, F: also inflationary.

Phase space behavior:

- Orbits approach H, inflate, then transition to E.
- Phantom divide crossing possible: $w = -1$.

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Quintom Dynamics and Phase Portrait

Model Features

- Dual fields: Quintessence ($w > -1$), Phantom ($w < -1$)
- Transitions across phantom divide ($w = -1$)
- Bouncing cosmologies and multiple inflationary epochs
- Compact phase space with critical point classification

Phase Portrait Highlights

- Saddle points: G, H (bouncing and inflationary behavior)
- Sinks: E (late-time inflation), F (graceful exit)
- Time-symmetric orbit through origin: bouncing solution

Phase Portrait of the Dynamical System

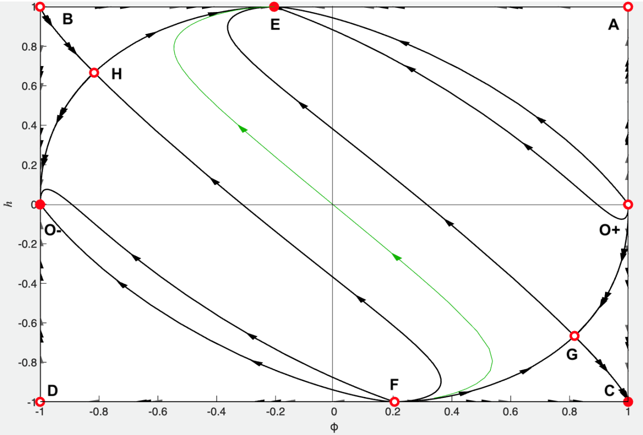
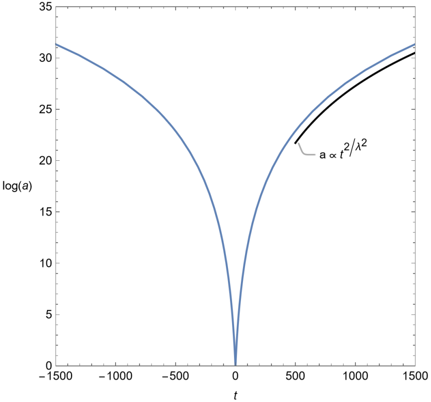


Figure: Orbits within the phase-space $[-1, 1]^2$, for $\lambda = 0.5$, $\kappa = 1.5$. The stable and unstable manifolds of the saddles G and H are shown, as well as orbits tangent to the eigenvectors of nodes (sinks E, O_+, C, and sources F, O_-, B). The green solution passing through the origin is time-symmetric and represents a bouncing cosmology.

Bouncing Cosmology



This figure shows the scale factor (logarithmically) vs proper time for the solution of (1)-(2) with initial conditions $\Phi = h = 0$ at time $t = 0$, for parameter values $\lambda = 0.2$, $\kappa = 0.61$. This corresponds to the green orbit shown before, and represents a bouncing cosmology. The late-time accelerating expansion is $a \propto t^{\lambda^{-2}/3}$.

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- Time-symmetric orbit through origin: bouncing solution

Mnemonic Tags: Homer (chaotic), Marge (stable), Bart (oscillatory), Burns (slow-roll), Flanders (exit)

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Inflationary Dynamics with Spatial Curvature ($K \neq 0$)

Extended system with curvature:

$$\Phi' = -\frac{(1-\Phi^2)}{\sqrt{2}}\left(\sqrt{6}h\Phi + \kappa(1 - h^2 - K\Omega_K) + \lambda\right)$$

$$h' = \sqrt{3}\left[(1 - h^2 - K\Omega_K)\left(1 + \frac{\kappa h\Phi}{\sqrt{6}}\right) - \Phi^2(1 - h^2) + \frac{1}{3}K\Omega_K\right]$$

$$\Omega_K' = -2\Omega_K\left[\frac{h}{\sqrt{3}} - \frac{\Phi}{\sqrt{2}}\left(\sqrt{6}h\Phi + \kappa(1 - h^2 - K\Omega_K)\right)\right]$$

Inflationary fixed points (with character tags):

$\tilde{E}$ (Mr. Burns) : $\left(-\frac{\lambda}{\sqrt{6}}, 1, 0\right)$, $\tilde{F}$ (Ned Flanders) : $\left(\frac{\lambda}{\sqrt{6}}, -1, 0\right)$

$\tilde{G}$ (Professor Frink) : $\left(\frac{\sqrt{6}}{\sqrt{\kappa^2+\kappa\lambda+6}}, -\frac{\kappa+\lambda}{\sqrt{\kappa^2+\kappa\lambda+6}}, 0\right)$

$\tilde{H}$ (Sideshow Bob) : $\left(-\frac{\sqrt{6}}{\sqrt{\kappa^2+\kappa\lambda+6}}, \frac{\kappa+\lambda}{\sqrt{\kappa^2+\kappa\lambda+6}}, 0\right)$

Inflation condition:

$$q = \frac{3}{h^2}\left(h^2 + \Phi^2 + \frac{2}{3}K\Omega_K - 1\right) - 1 < 0$$

Extension with Spatial Curvature

- Introduce Ω_K as a third variable in the dynamical system.
- Evolution equations for Φ and h are modified.
- Inflationary fixed points persist: $\tilde{E}, \tilde{F}, \tilde{G}, \tilde{H}$.
- Eigenvalues now depend explicitly on curvature.

Curvature $\Rightarrow$ geometric deformation of FLRW background.

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Why Is Curvature Required?

- **Physical motivation:** early-universe scenarios may involve $K \neq 0$.
- **Inflation condition modified:** deceleration parameter q includes curvature:

$$q = \frac{3}{h^2}\left(h^2 + \Phi^2 + \tfrac{2}{3}K\Omega_K - 1\right) - 1$$

- **Dynamical richness:** new invariant sets and bifurcation structures emerge.

Curvature $\Rightarrow$ more routes, more transitions, more physics.

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Simpsons Analogy

- **Mr. Burns** ($\tilde{E}$): still drives inflation, now with geometric resistance.
- **Ned Flanders** ($\tilde{F}$): graceful exit, curvature may help or hinder.
- **Prof. Frink** ($\tilde{G}$): saddle behavior becomes more chaotic with Ω_K.
- **Sideshow Bob** ($\tilde{H}$): elegant instability, now curvature-tuned.

Curvature $\Rightarrow$ Springfield on a warped donut.

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Conclusion

- Curvature is not a technical add-on — it's essential for full dynamical capture.
- Enables richer transitions and more complicated structures.
- In the analogy, Springfield gains depth, and characters reveal new roles.

Curvature $\Rightarrow$ key ingredient in scalar cosmology.

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Linear Perturbations: Setup

Perturbed FLRW metric (Newtonian gauge):

$$ds^2 = a^2(\sigma)\left[-(1+2\mathcal{S})d\sigma^2 + (1-2\mathcal{S})\gamma_{ij}dx^i dx^j\right]$$

$$\frac{d}{d\sigma} = a\frac{d}{dt}, \quad \mathcal{H} = \frac{a'}{a} = aH$$

Field perturbations:

$$\tilde{\varphi}^A = \varphi^A + \delta\varphi^A, \quad h_{AB}(\tilde{\varphi}) = h_{AB} + \partial_C h_{AB}\delta\varphi^C$$

Einstein equations (perturbed):

$$\mathcal{S}'' + \nabla^2\mathcal{S} + 4K\mathcal{S} = \varphi'\phi' - e^{\kappa\phi}\xi'\psi' - \tfrac{1}{2}\kappa e^{\kappa\phi}\psi'^2\varphi$$

$$\mathcal{H}\mathcal{S} + \mathcal{S}' = \tfrac{1}{2}\varphi\phi' - \tfrac{1}{2}e^{\kappa\phi}\xi\psi'$$

Decomposition:

$$\mathcal{S} = \Xi + \Sigma, \quad \Xi \leftrightarrow \phi, \quad \Sigma \leftrightarrow \psi$$

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Perturbation Dynamics & Reduction

Fourier modes:

$$\Xi'' - k^2\Xi = \phi'\varphi', \quad \Sigma'' - k^2\Sigma = -e^{\kappa\phi}\xi'\psi' - \tfrac{1}{2}\kappa e^{\kappa\phi}\psi'^2\varphi$$

Time reparametrization:

$$d\tau = \chi dt, \quad Z = \frac{k^2}{(a\chi)^2}$$

Perturbation definitions:

$$\varphi = \frac{\sqrt{2}(\sqrt{3}h\Xi + 3\Xi')}{3\Phi}, \quad \zeta = -\frac{\sqrt{2}(\sqrt{3}h\Sigma + 3\Sigma')}{3\sqrt{1-h^2}}$$

Reduced equation for Ξ:

$$\Xi'' = \Xi\left(-3Z + \frac{F(\Phi, h, \kappa, \lambda)}{\Phi}\right) + \Xi' G(\Phi, h, \kappa, \lambda)$$

Phase reduction:

$$\Xi = f_1 + if_2, \quad f = r\cos\theta, \quad f' = r\sin\theta \Rightarrow \theta' = -\sin^2\theta - P\sin\theta\cos\theta - Q\cos^2\theta$$

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Extended Phase Space Summary

State space: $S = B \times P$ with $(\Phi, h) \in B$, $(\bar{Z}, \theta) \in P$

Compact variable:

$$\bar{Z} = \frac{Z}{1+Z} = \frac{k^2}{k^2 + (a\chi)^2}, \quad d\bar{\tau}/d\tau = 1 + Z = (1 - \bar{Z})^{-1}$$

Autonomous system:

$$\begin{aligned} \Phi' &= \tfrac{1}{2}(1-\Phi^2)(1-\bar{Z})F_1(\Phi, h) \\ h' &= \tfrac{1}{2}(1-h^2)(1-\bar{Z})F_2(\Phi, h) \\ \bar{Z}' &= \tfrac{1}{3}(1-\bar{Z})^2 \bar{Z} F_3(\Phi, h) \\ \theta' &= \text{Trigonometric expression in } (\Phi, h, \bar{Z}, \theta) \end{aligned}$$

Critical points:

- Basic: A–H, $O_\pm$; Rotated: A^*–H^*; Invariant set $\bar{Z} = 1$: $\bar{A}$–$\bar{H}$

Angular dynamics on $\bar{Z} = 1 : \theta' = \cos^2(\theta)$, modulo $n\pi$

- Homer (chaotic source): A, B, $\bar{C}$
- Frink (saddle with complicated dynamics): $\bar{D}$, F^*
- Sideshow Bob (elegant instability): $\bar{F}$
- Marge (stabilizing influence): $\bar{A}$, $\bar{B}$
- Bart (oscillatory behavior): C, D
- Mr. Burns (slow-roll inflation): E^*
- Flanders (graceful exit): $\bar{E}$

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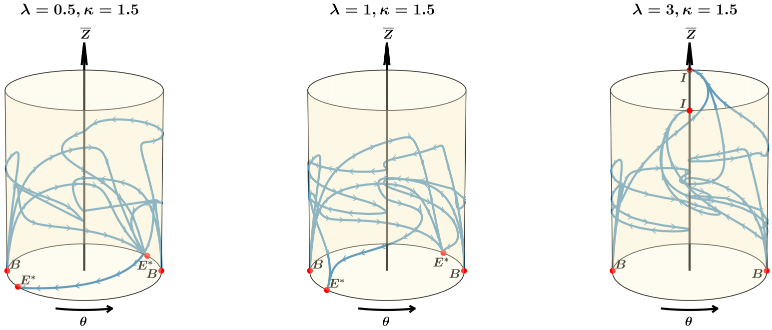
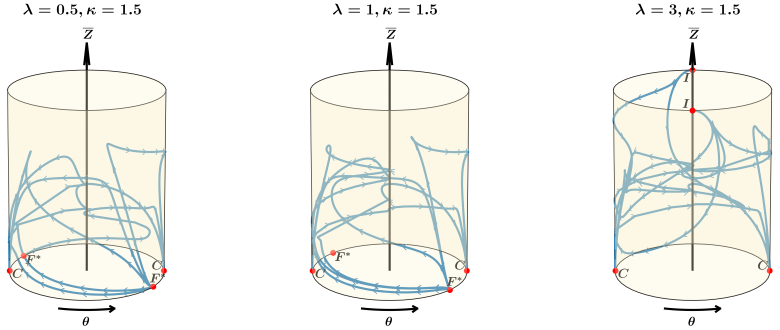
Projections of Solutions (Perturbations)



(a) Projections $(\cos\theta, \sin\theta, \bar{Z})$ for $h = 1$

Figure: Projections of solutions of the system describing perturbations. The figures illustrate the behavior of the angular variable θ and the compactified variable $\bar{Z}$.

Projections of Solutions (Perturbations)



(a) Projections $(\cos\theta, \sin\theta, \bar{Z})$ for $h = -1$

Figure: Projections of solutions of the system describing perturbations. The figures illustrate the behavior of the angular variable θ and the compactified variable $\bar{Z}$.

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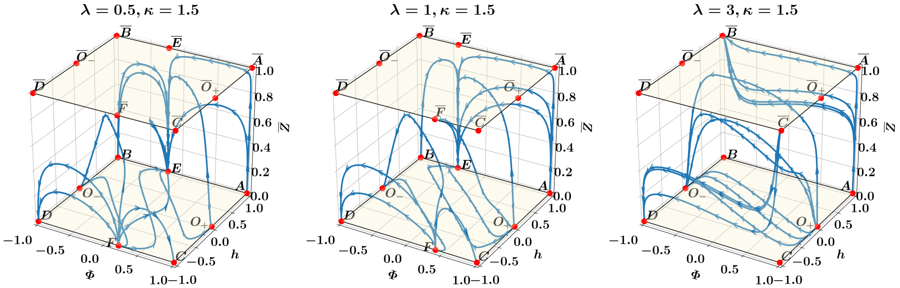
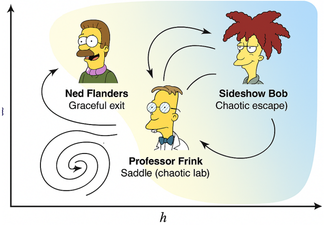
Projections of Solutions (Perturbations)



(a) Projections $(h, \Phi, \bar{Z})$ for various initial conditions

Figure: Projections of solutions of the system describing perturbations. The figures illustrate the behavior of the angular variable θ and the compactified variable $\bar{Z}$.

Phase Space Portraits



Character	Dynamical Role	Curvature Sensitivity
Mr. Burns	Inflationary sink	Stable under curvature
Ned Flanders	Graceful exit	Mildly affected
Professor Frink	Saddle (chaotic lab)	Strong sensitivity

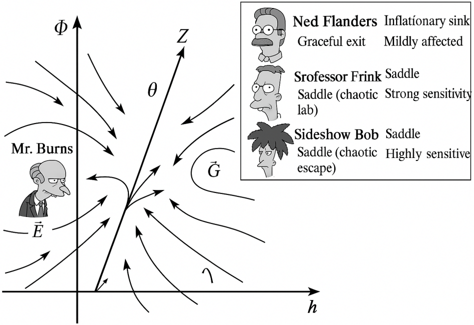


Figure: Left: Basic phase space. Right: With Z and θ perturbations.

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Conclusion: Including Linear Perturbations

- **Setup:** Perturbations are introduced over the curved FLRW background.
- Scalar modes evolve under modified equations due to Ω_K and dual fields.
- Stability of inflationary points $(\tilde{E}, \tilde{F})$ confirmed at linear level.
- Saddle points $(\tilde{G}, \tilde{H})$ show mode mixing and curvature-sensitive growth.

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Simpsons Analogy:

- **Mr. Burns** remains stable — even with Springfield shaking.
- **Flanders** absorbs perturbations gently — a graceful exit.
- **Frink** amplifies fluctuations — his lab explodes with curvature.
- **Bob** channels instability — perturbations follow his chaotic trail.

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- **Mr. Burns** remains stable — even with Springfield shaking.
- **Flanders** absorbs perturbations gently — a graceful exit.
- **Frink** amplifies fluctuations — his lab explodes with curvature.
- **Bob** channels instability — perturbations follow his chaotic trail.

Final Message: Linear perturbations validate the dynamical roles — curvature adds depth, and the cast performs on a richer stage.

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Summary and Future Directions

Key Results:

- Dynamical systems analysis of the quintom model.
- Multiple inflationary phases and phantom divide crossing.
- Inflationary solutions stable under spatial curvature.
- Bouncing cosmologies and structure formation via perturbations.

Next Steps:

- Generalize to broader classes of potentials.
- Include higher-order and non-linear perturbations.
- Constrain parameters with observational data.

Advances in Fractional Dynamics: Computational Approaches and Applications in Cosmology, Mechanics, and Complex Systems

Special Issue

Advances in Fractional Dynamics: Computational Approaches and Applications in Cosmology, Mechanics, and Complex Systems

Message from the Guest Editor

This Special Issue will highlight recent advances in the theoretical foundations and computational modeling of fractional dynamics, with a particular emphasis on applications in cosmology, classical and quantum mechanics, and the analysis of complex systems. Fractional calculus provides a powerful framework for capturing non-local behaviors, memory effects, and anomalous transport phenomena—features that frequently emerge in physical and mathematical models but remain inadequately addressed by classical differential approaches. We welcome the submission of original research contributions that advance analytical and numerical techniques for solving fractional differential equations, including novel computational algorithms, stability assessments, and geometric formulations. We will particularly focus on studies that demonstrate the applicability of these methods to gravitational systems, particle motion, and emergent structures in nonlinear and multifractal regimes.

Guest Editor

Dr. Genly Leon
Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta 1270709, Chile

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Editorial Office
MDPI, Grosspeteranlage 5
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Thank You!

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Questions?

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