Resonance-induced eccentricity in spherical EMRIs

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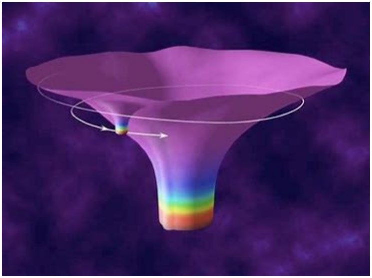


- EMRIs: binaries involving a stellar-mass compact object and a supermassive BH.
- Mass ratio: $\mu/M \sim 10^{-7} - 10^{-4}$.
- Gravitational waves: in the range $10^{-4} - 10^{-1}$ Hz, prime targets of LISA.
- Adiabatic approximation: the influence of gravitational back-reaction become significant on timescales much longer than any orbital timescale.
  - short-timescales: nearly geodesic motion
  - long-timescales: the system evolves through a sequence of geodesics

- We assume that the spacetime geometry of the central BH is described by the metric derived by Johannsen, [PRD **88** 044002 (2013)].
- In Boyer-Lindquist-like coordinates (t, r, θ, ϕ):

$$
\begin{aligned}
ds^2 = &- \frac{\tilde{\Sigma}[\Delta - a^2 A_2(r)^2 \sin^2\theta]}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2} dt^2 \\
&- 2a \frac{[(r^2+a^2)A_1(r)A_2(r) - \Delta]\tilde{\Sigma}\sin^2\theta}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2]^2} dt d\phi \\
&+ \frac{\tilde{\Sigma}}{\Delta A_5(r)} dr^2 + \tilde{\Sigma} d\theta^2 \\
&+ \frac{\tilde{\Sigma}\sin^2\theta[(r^2+a^2)^2 A_1(r)^2 - a^2\Delta\sin^2\theta]}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2} d\phi^2,
\end{aligned}
$$

$$
\Delta = r^2 + a^2 - 2Mr, \qquad \tilde{\Sigma} = r^2 + a^2\cos^2\theta + f(r).
$$

where a, M are still the spin and the mass of the BH.

The deviation functions A_1, A_2, A_5, f are given as a power series in M/r:

$$A_1(r) = 1 + \sum_{n=3}^{\infty} a_{1n} \left(\frac{M}{r}\right)^n, \qquad A_2(r) = 1 + \sum_{n=2}^{\infty} a_{2n} \left(\frac{M}{r}\right)^n,$$

$$A_5(r) = 1 + \sum_{n=2}^{\infty} a_{5n} \left(\frac{M}{r}\right)^n, \qquad f(r) = 0 + \sum_{n=3}^{\infty} \epsilon_n r^2 \left(\frac{M}{r}\right)^n.$$

- It reduces smoothly to the Kerr metric if $A_1(r) = A_2(r) = A_5(r) = 1$, and $f(r) = 0$.
- The metric is stationary, axisymmetric, reflection symmetric along the equatorial plane and asymptotically flat, but not a vaccuum solution.
- It is actually not a solution to the field equations of any particular gravity theory. It is simply an artificial variation of Kerr that keeps all its symmetries.

- The Johannsen metric possesses three independent and involution exact constants of motion (besides the rest mass μ), like Kerr:

$$
\begin{aligned}
E &= -p_t \\
L_z &= p_\phi \\
Q &= p_\theta^2 + \cos^2\theta \left[a^2(\mu^2 - E^2) + \frac{L_z^2}{\sin^2\theta} \right] \\
&= -\Delta p_r^2 A_5(r) + \frac{[(r^2+a^2)E A_1(r) - a L_z A_2(r)]^2}{\Delta} \\
&\quad - (L_z - aE)^2 - \mu^2[r^2 + f(r)].
\end{aligned}
$$

- The H-J equations are fully separable in all coordinates. Therefore we could formulate the equations of motion as 1st order diff. eqs. (like in Kerr):

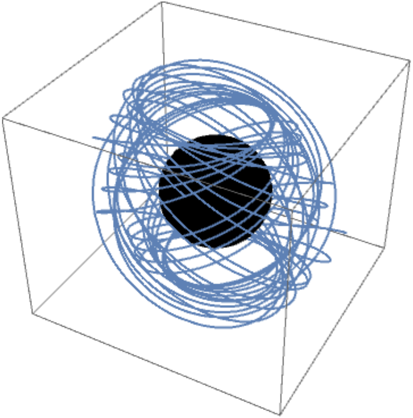
$$
\begin{aligned}
\mu\tilde{\Sigma}\frac{dr}{d\tau} &= \pm\sqrt{A_5(r)R(r)}, \\
\mu\tilde{\Sigma}\frac{d\theta}{d\tau} &= \pm\sqrt{\Theta(\theta)},
\end{aligned}
$$

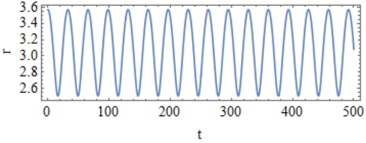
where

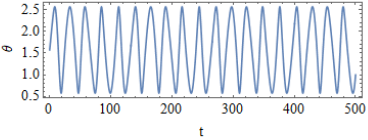
$$
R(r) = [E(r^2+a^2)A_1(r) - a L_z A_2(r)]^2 - \Delta(\mu^2[r^2+f(r)] + [L_z - aE]^2 + Q],
$$

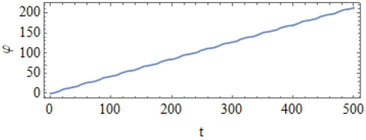
$$
\Theta(\theta) = Q - \cos^2\theta \left[a^2(\mu^2 - E^2) + \frac{L_z^2}{\sin^2\theta} \right].
$$

- $a = 0.85$, $M = 1$, $a_{13} = -1.5$, $a_{22} = 4$, $a_{52} = 6$, and $\epsilon_3 = 4$.









Fundamental frequencies [**AE** et al,PRD 110.124004]

By analyzing the geodesic dynamics in action-angle variables we obtain the 3 fundamental frequencies:

$$\Omega_r = \frac{\pi K(k)}{a^2 z_+ [K(k) - E(k)] X + Y K(k)}$$

for radial oscillations (eccentric orbits)

$$\Omega_\theta = \frac{\pi \beta \sqrt{z_+} X/2}{a^2 z_+ [K(k) - E(k)] X + Y K(k)}$$

for oscillations about the equator (non-planar orbits)

$$\Omega_\phi = \frac{Z K(k) + X L_z [\Pi(z_-, k) - K(k)]}{a^2 z_+ [K(k) - E(k)] X + Y K(k)}$$

rotations around the axis of symmetry (orbital winding).

...the elliptic integrals involved

The $K(k)$, $E(k)$ and $\Pi(z_-, k)$ are the 1st, 2nd and 3rd complete elliptic integrals:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta,$$

$$\Pi(z_-, k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - z_- \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}},$$

with $z = \cos^2\theta$, $k = \sqrt{z_-/z_+}$ (where $z_\pm$ are the two roots of $\Theta(z) = 0$ with $0 < z_- = \cos^2\theta_{min} < 1 < z_+$), while $\beta^2 = a^2(1 - E^2)$.

The Y, Z and X are the radial integrals:

$$Y = \int_{r_2}^{r_1} \frac{r^2 + f(r)}{\sqrt{A_5(r)R(r)}} dr$$

$$Z = \int_{r_2}^{r_1} \frac{aA_2(r)((r^2 + a^2)A_1(r)E - aA_2(r)L_z) + \Delta(L_z - aE)}{\Delta\sqrt{A_5(r)R(r)}} dr$$

$$X = \int_{r_2}^{r_1} \frac{dr}{\sqrt{A_5(r)R(r)}},$$

where r_1(apastron) and r_2(periastron) are the turning points of the radial motion, i.e. $R(r_1) = R(r_2) = 0$.

Kennefick and Ori argument [Phys.Rev.D 53 (1996) 4319-4326]

- The rate of change of Q has the "appropriate" form so that spherical orbits remain spherical at adiabatic order so long as the self-force does not resonate with the radial oscillations.
- At the adiabatic limit, radiation reaction drives a particle in spherical motion around a Kerr BH through successively damped spherical geodesics.
- The periodicity of the GSF for a spherical orbit is determined by the polar motion and due to the reflection symmetry of the metric, the frequency of GSF is twice the polar frequency. Thus the only assumption for this stability to hold is the fact that there is no resonance of the form

$$\frac{\Omega_r}{\Omega_\theta} = 2n, \quad n = 1, 2, \ldots$$

  for some integer n.
- In Kerr case the resonance condition is never met, for any generic orbit.

$$\frac{\Omega_r}{\Omega_\theta} < 1.$$

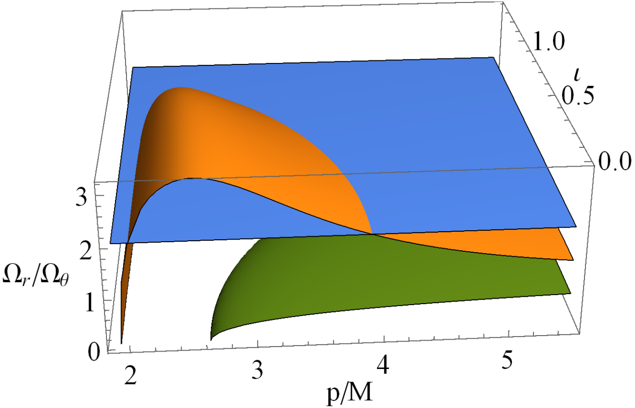
Resonance Condition

- Resonance condition: $\Omega_r/\Omega_\theta = 2$.
- This is satisfied when: $\beta\sqrt{z_+}X = K(k)$,
- Deviation functions:

$$A_1(r) = 1 + a_{13}\frac{M^3}{r^3}, \quad A_2(r) \quad = 1 + a_{22}\frac{M^2}{r^2}, \quad f(r) = \epsilon_3\frac{M^3}{r} \qquad \text{and}$$

$$A_5(r) = 1 + a_{52}\frac{M^2}{r^2},$$

- eccentricity: $e = \frac{r_1 - r_2}{r_1 + r_2}$



- $a = 0.85M$, $\mu/M = 10^{-6}$, $a_{13} = -1.5$, $a_{22} = 4$, $a_{52} = 6$ and $\epsilon_3 = 4$.
- The volume of the parameter space where bound geodesic motion occurs can be significantly larger. Consequently, the resonant condition could be satisfied.
- The resonance $\Omega_r/\Omega_\theta = 2$ is present in this specific Johannsen metric.

- The orbital evolution is computed, based on the average losses due to radiation reaction, since we do not have an exact formulae for the GSF in non Kerr spacetimes.
- Average losses of E and L_z are computed from the hybrid kludge scheme (Gair & Glampedakis, 2006) which combines exact expressions for the evolution of the orbital elements with a second order Post-Newtonian radiation reaction formulae for the “constants” of motion fluxes.
- We integrate the geodesic equations

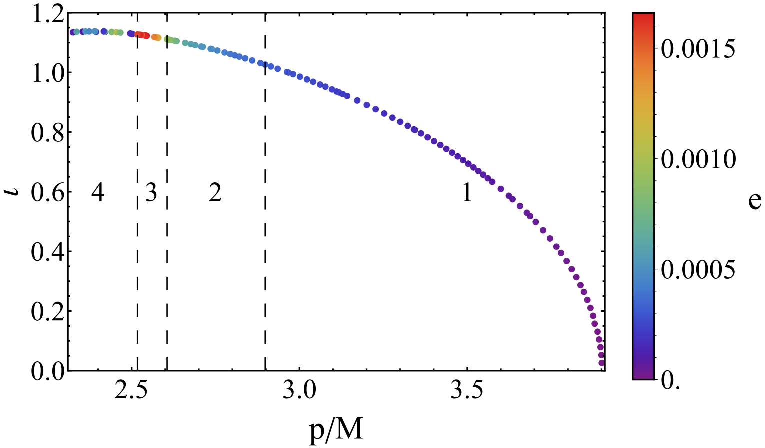
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\kappa_{\lambda\nu}(E, L_z)\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

for small interval of time compared to the radiation reaction timescale, assuming a linear evolution of E and L_z:

$$E(t) = E_0 + \langle \frac{dE}{dt}\rangle\Big|_0 t,$$

$$L_z(t) = L_{z,0} + \langle \frac{dL_z}{dt}\rangle\Big|_0 t.$$

$a = 0.85M$, $\mu/M = 10^{-6}$, $a_{13} = -1.5$, $a_{22} = 4$, $a_{52} = 6$ and $\epsilon_3 = 4$.



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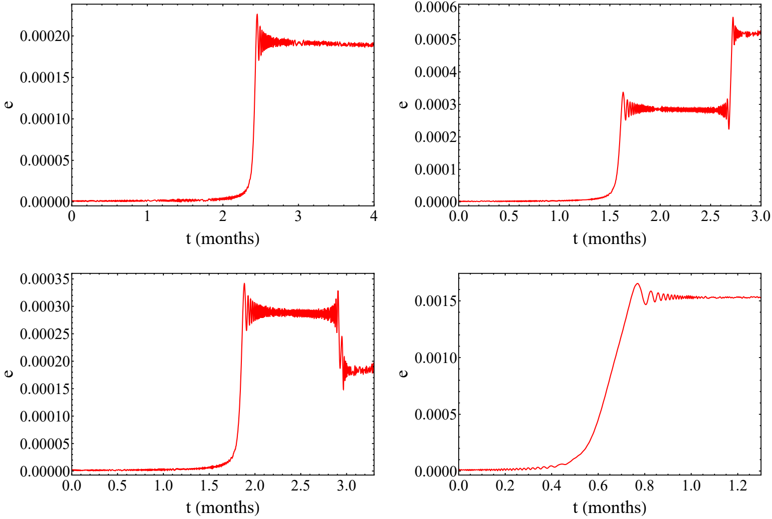


By adding the deformation parameter a_Q [**KD** et al. PRD **102**, 064041 (2020)], that breaks the integrability of geodesics, the Johannsen metric becomes:

$$
\begin{aligned}
ds^2 = &- \frac{\widetilde{\Sigma}[M^3(a_Q/r) + \Delta - a^2 A_2(r)^2 \sin^2\theta]}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2} dt^2 \\
&- 2a \frac{[(r^2+a^2)A_1(r)A_2(r) - \Delta]\widetilde{\Sigma}\sin^2\theta}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2]^2} dt d\phi \\
&+ \frac{M^3(a_Q/r) + \widetilde{\Sigma}}{\Delta A_5(r)} dr^2 + \widetilde{\Sigma} d\theta^2 \\
&+ \frac{\widetilde{\Sigma}\sin^2\theta[(r^2+a^2)^2 A_1(r)^2 - a^2\Delta\sin^2\theta]}{[(r^2+a^2)A_1(r) - a^2 A_2(r)\sin^2\theta]^2} d\phi^2,
\end{aligned}
$$

- $a = 0.85$, $\mu/M = 10^{-6}$, $a_{13} = -1.5$, $a_{22} = 4$, $a_{52} = 6$, and $\epsilon_3 = 4$.

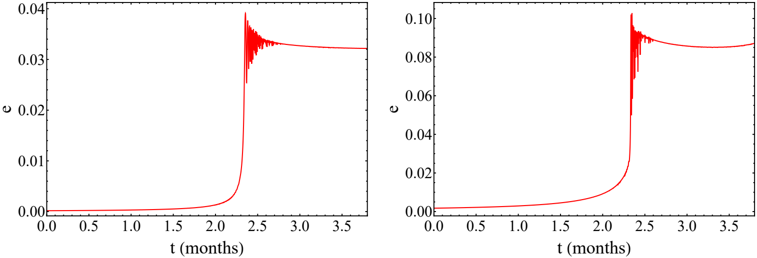
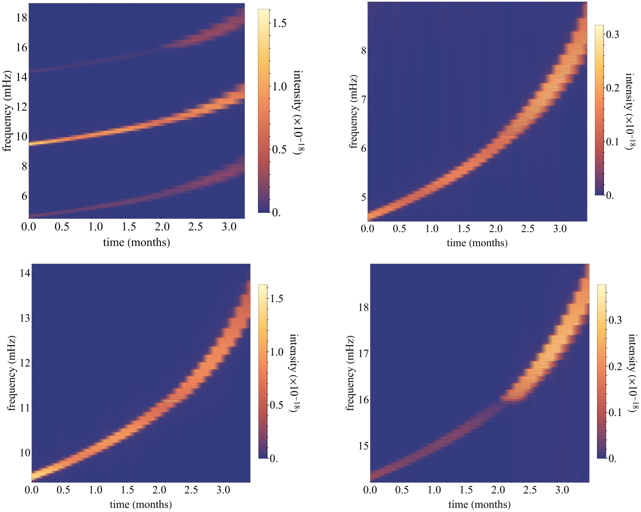


Figure 1: left fig. $a_Q = 10^{-3}$, right fig $a_Q = 10^{-2}$

- Waveform modelling: We use the numerical kludge scheme that combines exact particle trajectories with approximate GW radiation emission [Babak et al., PRD 75, 024005 (2007)].
- Quadrupole formula: $h_{ij}^{TT} = \frac{2}{D} \frac{d^2 I_{ij}^{(STF)}}{dt^2}$.
- GW components: $h_{+,\times} = \frac{2\mu}{D} \epsilon_{ij}^{+,\times} \left[\frac{d^2 \mathbf{Z}^i(t)}{dt^2} \mathbf{Z}^j(t) + \frac{d\mathbf{Z}^i(t)}{dt} \frac{d\mathbf{Z}^j(t)}{dt} \right]$,
- Total GW waveform detected by LISA: $h_a = \frac{\sqrt{3}}{2} [F_a^+(t) h_+(t) + F_a^-(t) h_-(t)$, where $a = \{I, II\}$ the channel indices of the detector's antenna.
- Assumptions:
  - the noise is stationary and Gaussian with zero mean.
  - The two data streams sectors are uncorrelated.
  - The noise power spectral density of LISA $S_n(f)$ is equivalent at both channels.


frequency (mHz)
time (months)
intensity (×10⁻¹⁸)
frequency (mHz)
time (months)
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intensity (×10⁻¹⁸)
frequency (mHz)
time (months)
intensity (×10⁻¹⁸)

Conclusions

- The Kennefick-Ori statement is verified in practice.
- The eccentricity of an orbit (in such a non-Kerr metric) could rise abruptly during the evolution of the orbit.
- However, this rise is small (at least for the Johannsen metric we used). It is of order 10^{-3} at most.
- The passage through a resonance could be double, and the second change of eccentricity is sometimes negative.
- We have checked that the change in eccentricity, when the initial eccentricity is not 0, is still small.
- The frequency evolution of an incoming GW contains a clear imprint of eccentricity excitation when the $\Omega_r/\Omega_\theta = 2$ resonance is crossed by an initially spherical orbit.