"Recent Developments in Gravity" (NEB-21), Corfu

"On the geometric origin of the energy-momentum tensor improvement terms"

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The talk is based on the paper: **"On the geometric origin of the energy-momentum tensor improvement terms"** (Damianos Iosifidis, Manthos Karydas, Anastasios Petkou, Konstantinos Siampos).
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Outline

- Brief review of some Field Theory results.
- Brief review of MAG and in particular its energy content
- Applications of our method to the Maxwell and Dirac fields in order to derive the improved stress tensors
- Application to the conformally invariant Scalar Field
- Further applications
- Conclusions/Future Projects

Field Theory

We consider matter fields $\phi^A(x)$, where A collectively denotes spacetime or internal symmetry indices, described by the action

$$S[\phi^A] = \int d^d x \, \mathcal{L}(\phi^A, \partial_\mu \phi^A) \,. \tag{1}$$

The metric is Minkowski with a mostly-plus signature. Let us consider general *active* coordinate transformations $x^\mu \mapsto x^\mu - \xi^\mu(x)$ that change the action by a total derivative. Under such transformations the fields change as

$$\phi^A \mapsto \phi^A + \delta_\xi \phi^A = \phi^A + \xi^\mu \partial_\mu \phi^A + \Delta_\xi \phi^A \,. \tag{2}$$

We split $\delta_\xi \phi^A$ into a first term that it is always there and a second term, $\Delta_\xi \phi^A$, that depends on the tensorial properties of ϕ^A as well as on the form of the transformation.

Conserved Currents

Then, provided that the equations of motion hold, we find the conserved currents (for a spacetime symmetry in flat space)

$$\partial_\mu \left(\xi^\nu t^\mu{}_\nu + \pi^{\mu,A} \Delta_\xi \phi^A \right) = 0 \tag{3}$$

where the canonical e.m. tensor $t^\mu{}_\nu$ has been defined as

$$t^\mu{}_\nu = \pi^{\mu,A} \partial_\nu \phi^A - \delta^\mu_\nu \mathcal{L} \, , \quad \pi^{\mu,A} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \, , \tag{4}$$

From (3) we can read the conditions satisfied by the canonical e.m. tensor t^μ_ν which lead to the improvement process.

Conservations

- Translations correspond to $\xi^\mu = a^\mu$ with a^μ constant. This gives

$$\Delta_a \phi^A = 0 \;\Rightarrow\; \partial_\mu t^\mu_{\ \nu} = 0\,. \tag{5}$$

- Lorentz rotations correspond to

$$\xi^\mu = \omega^\mu{}_\nu x^\nu\,, \quad \omega^{\mu\nu} = -\omega^{\nu\mu} = \text{constants}\,. \tag{6}$$

This gives

$$\Delta_\omega \phi^A = \omega_{\mu\nu}\Sigma^{\mu\nu,A}\,, \tag{7}$$

where $\Sigma^{\mu\nu,A} = -\Sigma^{\nu\mu,A}$ is a function of the fields ϕ^A that depends on their Lorentz properties. Substituting (7) into (3) we obtain

$$t^{\mu\nu} - t^{\nu\mu} = 2\partial_\rho \Psi^{\rho[\mu\nu]}\,, \tag{8}$$

where the quantity $\Psi^{\rho[\mu\nu]} = \pi^{\rho,A}\Sigma^{[\mu\nu],A}$ is referred to as the *spin tensor*.

The last immediately leads to the well-known Belinfante-Rosenfeld (BR) improvement procedure

$$t^{\mu\nu} = T^{\mu\nu}_{\text{BR}} + \partial_\rho \left(\Psi^{\mu[\nu\rho]} + \Psi^{\nu[\mu\rho]} + \Psi^{\rho[\mu\nu]} \right) , \tag{9}$$

where $T^{\mu\nu}_{\text{BR}}$ is the 'improved' energy-momentum tensor

$$\partial_\mu T^{\mu\nu}_{\text{BR}} = 0 \, , \quad T^{\mu\nu}_{\text{BR}} = T^{\nu\mu}_{\text{BR}} \, . \tag{10}$$

Metric-Affine Geometry

Two distinctively different notions on a manifold

- Metric Tensor $g_{\mu\nu}$: Defines distances,lengths and dot products

$$||\alpha||^2 := \alpha^\mu \alpha^\nu g_{\mu\nu} \ , \ \ (\alpha \cdot \beta) := \alpha^\mu \beta^\nu g_{\mu\nu}$$

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$$\nabla_\lambda u^\mu = \partial_\lambda u^\mu + \Gamma^\mu{}_{\nu\lambda} u^\nu$$

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Non-Riemannian geometry:

A generalized geometry where no a priori relation between the connection and the metric is assumed. The connection is in general neither symmetric nor metric-compatible.

Definitions of Torsion, Curvature and Non-metricity

Torsion

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \tag{11}$$

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Non-metricity

$$Q(Z,X,Y) := -\nabla_Z g(X,Y) \tag{13}$$

where X, Y, Z are C^∞ vector fields.

Geometrical Objects in Component Form

Torsion

- $\nabla_{[\mu}\nabla_{\nu]}\phi = S_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi$, Torsion Tensor $S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}$

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Curvature

- $[\nabla_{\alpha}, \nabla_{\beta}]u^{\mu} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu} + 2S_{\alpha\beta}{}^{\nu}\nabla_{\nu}u^{\mu}$
  Curvature Tensor: $R^{\mu}{}_{\nu\alpha\beta} := 2\partial_{[\alpha}\Gamma^{\mu}{}_{|\nu|\beta]} + 2\Gamma^{\mu}{}_{\rho[\alpha}\Gamma^{\rho}{}_{|\nu|\beta]}$

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Nonmetricity

- $Q_{\alpha\mu\nu} := -\nabla_{\alpha}g_{\mu\nu} = -\partial_{\alpha}g_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\nu} + \Gamma^{\lambda}{}_{\nu\alpha}g_{\lambda\mu}$

Torsion/Nonmetricity related vectors

$$S_\mu = S_{\mu\lambda}{}^{\lambda} \, , \quad t^\mu = \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho\sigma} \;\; (only \;\; for \;\; n = 4)$$

$$Q_\mu = g^{\alpha\beta} Q_{\mu\alpha\beta} \, , \qquad q_\mu = g^{\rho\alpha} Q_{\rho\alpha\mu}$$

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Notation

The general affine-connection will be denoted by ∇ whereas the (Riemannian) Levi-Civita connection will read $\tilde{\nabla}$. All quantities with a tilde will denote Riemannian parts (i.e. computed wrt $\tilde{\nabla}$).

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Remark

In the following the presence of non-metricity will not be essential so we may set $Q_{\alpha\mu\nu} = 0$.

Metric-Affine Gravity

Metric Gravity

- $\Gamma^{\alpha}{}_{\mu\nu} \rightarrow$ *torsionless* , metric compatibility $\nabla_{\sigma} g_{\mu\nu} = 0$
- $S = S_{Gravity} + S_{Matter} = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Phi) \right]$

Metric-Affine Gravity (MAG)

- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}, \Phi) \right] \Rightarrow$ No a priori constraints on the geometry.
- Field Equations

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \;, \quad \frac{\delta S}{\delta \Gamma^{\lambda}{}_{\mu\nu}} = 0 \tag{14}$$

The Sources: Canonical, Metrical and Hypermomentum Energy Momentum Tensors

Canonical (Noether) Energy-Momentum Tensor

$$\Sigma^{\mu}{}_{\nu} := \frac{\partial \mathcal{L}_M}{\partial(\nabla_\mu \psi^A)} \nabla_\nu \psi^A - \delta^\mu_\nu \mathcal{L}_M \tag{15}$$

Metrical (Hilbert) Energy Momentum-Tensor

$$T_{\alpha\beta} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\alpha\beta}} \tag{16}$$

Hypermomentum Tensor

$$\Delta_{\lambda}{}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}} \tag{17}$$

A note on the importance of Hypermomentum

Decomposition

The Hypermomentum tensor can be split into its three irreducible pieces (in exterior calculus!) of spin, dilation and shear according to (see [Hehl et al, 1995])

$$\Delta_{\alpha\mu\nu} = \Sigma_{\alpha\mu\nu} + \frac{1}{n} g_{\alpha\mu}\Delta_{\nu} + \hat{\Delta}_{\alpha\mu\nu} \tag{18}$$

with

$$\sigma^{\mu\nu\alpha} := \Delta^{[\mu\nu]\alpha} \quad (Spin) \tag{19}$$

$$\Delta^{\nu} := \Delta^{\alpha\mu\nu} g_{\alpha\mu} \quad (Dilation) \tag{20}$$

$$\Sigma^{\mu\nu\alpha} := \Delta^{(\mu\nu)\alpha} - \frac{1}{n} g^{\mu\nu}\Delta^{\alpha} \quad (Shear) \tag{21}$$

The Hypermomentum describes the intrinsic properties of matter!

Connection with the 'Imperfection' of Relativistic Fluids

It was recently shown [**DI** & Tomi Koivisto, JCAP 05 (2024) 001] that the hypermomentum is actually associated to the viscous properties of relativistic fluids. In particular bulk viscosity, heat flux etc. arise as certain parts of hypermomentum. Furthermore:

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- In contrast to the standard first order hydrodynamics formulation where the dynamical eqns for the viscous properties must be imposed by hand here they arise naturally from the conservation laws of MAG.
- This follows from the Lagrangian formulation of fluids with connection-matter couplings. The hypermomentum of such fluids is isotropic but results in an imperfect energy-momentum tensor!

Conservation Laws

Working in exterior calculus from the GL and diff invariance we get

From GL(n,ℝ)

$$\Sigma^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} + \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}) \ , \quad \hat{\nabla}_{\mu} = \nabla_{\mu} - 2S_{\mu}$$

From Diff

$$-\frac{1}{\sqrt{-g}}\hat{\nabla}_{\mu}(\sqrt{-g}\Sigma^{\mu}{}_{\alpha}) = -\frac{1}{2}\Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha} + \frac{1}{2}Q_{\alpha\mu\nu}T^{\mu\nu} + 2S_{\alpha\mu\nu}\Sigma^{\mu\nu}$$

From Diff using coordinates

$$\sqrt{-g}(2\tilde{\nabla}_{\mu}T^{\mu}{}_{\alpha} - \Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha}) + \hat{\nabla}_{\mu}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\alpha}{}^{\mu\nu})$$
$$-2S_{\mu\alpha}{}^{\lambda}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}) = 0$$

Flat Space limit ($g = \eta$, $\Gamma = 0$) of Conservation Laws

Idea: Let us take the flat space (Minkowski) limit of the previous conservation laws. They read

$$\boxed{t^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \Delta^{\nu\mu\lambda}\,, \quad \partial_\mu t^{\mu\nu} = 0\,.} \tag{22}$$

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Note that $\Sigma_{\mu\nu} \to t_{\mu\nu}$ in the flat space limit.

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Hypermomentum as the improvement term

It is important to stress that in general only $t^{\mu\nu}$ is conserved. However if $\Delta_{\alpha\mu\nu} = \Delta_{\alpha[\mu\nu]}$ then $T^{\mu\nu}$ is ensured to be conserved! This happens when the connection dependence comes solely from the torsion tensor. This is the case in all of our examples.

The action for the free Maxwell field is

$$S_{\text{EM}} = \int d^d x \mathcal{L}_{\text{EM}}\,, \quad \mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{23}$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the electromagnetic field. The action is invariant under gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu f$ and its equations of motion are given by

$$\partial_\lambda F^{\mu\lambda} = 0\,. \tag{24}$$

The corresponding canonical energy–momentum tensor is found to be

$$t^{\mu\nu} = F^{\mu\lambda} \partial^\nu A_\lambda - \frac{1}{4} \eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}\,. \tag{25}$$

This is neither symmetric nor gauge invariant. To obtain an improved tensor that meets these requirements one uses the BR method, which amounts to adding the term [Landau & Lifshitz]

$$- F^{\mu\lambda} \partial_\lambda A^\nu\,. \tag{26}$$

The inclusion of the latter gives the symmetric and also gauge invariant electromagnetic energy–momentum tensor

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \,. \tag{27}$$

The latter can be alternatively obtained if we minimally couple the action (23) to Einstein gravity as

$$S_{\text{EM}} = -\frac{1}{4}\int d^d x \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{28}$$

and then use (16). The resulting Hilbert metrical e.m. tensor coincides in the flat limit with (27). However, as we have mentioned above the minimal coupling of the theory to Einstein gravity obscures the access to the canonical e.m. tensor. This is rectified by coupling the theory to MAG as follows.

Prescription.

Consider the minimal substitution $\partial_\mu \to \nabla_\mu$. Then the field strength becomes

$$\hat{F}_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu = F_{\mu\nu} + 2S_{\mu\nu}{}^{\lambda} A_\lambda \,, \tag{29}$$

which is not gauge invariant[a] and the corresponding Lagrangian becomes

$$\hat{\mathcal{L}}_{\text{EM}} = -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = -\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} + 4S_{\mu\nu}{}^{\rho}F^{\mu\nu}A_\rho + 4S_{\mu\nu}{}^{\rho}S^{\mu\nu\lambda}A_\rho A_\lambda\right) . \tag{30}$$

[a]The breaking of gauge invariance is not fatal at this point as this is only an intermediate step. At the end we will go back to the flat background (i.e. we set $g = \eta$ and $\Gamma^{\lambda}{}_{\mu\nu} = 0$) and the gauge invariance will then be restored.

Prescription.

For this Lagrangian compute the associated hypermomentum and consequently take the flat space limit, to find

$$\Delta_{\lambda}{}^{\mu\nu} = 2A_{\lambda}F^{\mu\nu}\,. \tag{31}$$

Then, using the equations of motion (24) we obtain

$$\frac{1}{2}\partial_{\lambda}\Delta^{\nu\mu\lambda} = F^{\mu\lambda}\partial_{\lambda}A^{\nu}\,, \tag{32}$$

quite miraculously this is exactly the term we wanted! Therefore, hypermomentum provides a geometric origin for the BR improvement term, as

$$T_{\mu\nu} = t_{\mu\nu} - \frac{1}{2}\partial_{\lambda}\Delta^{\nu\mu\lambda} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}\,. \tag{33}$$

Extension to p-forms

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Was this just a coincidence or is the prescription universal? Let us study also the Dirac field and see!

Dirac Field in Flat Space

$$S_{\mathrm{D}} = \int d^d x \left[-\frac{i}{2}\left(\bar{\psi}\partial\!\!\!/\psi - (\partial\!\!\!/\bar{\psi})\psi \right) - im\bar{\psi}\psi \right], \qquad (34)$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$.

Energy-momentum tensor

The canonical energy-momentum tensor is then found to be

$$t^{\mu\nu} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi \right) . \qquad (35)$$

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Let us apply our prescription and see what do we get.

Going Affine

$$S_{\mathrm{D}} = \int d^d x \sqrt{-g}\mathcal{L}_{\mathrm{D}}\,, \quad \mathcal{L}_{\mathrm{D}} = -\frac{i}{2}\left(\bar{\psi}\gamma^c \nabla_c \psi - \nabla_c \bar{\psi}\gamma^c \psi\right) - im\bar{\psi}\psi\,, \tag{36}$$

$\gamma' s$ obey Clifford algebra $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$, and $(\gamma^a)^\dagger = \gamma^0\gamma^a\gamma^0$. Furthermore, we have the useful identity

$$[\gamma^a, \Sigma^{bc}] = \eta^{ab}\gamma^c - \eta^{ac}\gamma^b\,, \;\; \Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{37}$$

Most importantly

$$\nabla_c\psi = \partial_c\psi + \frac{1}{2}\omega_{abc}\Sigma^{ab}\psi\,, \quad \nabla_c\bar{\psi} = \partial_c\bar{\psi} - \frac{1}{2}\omega_{abc}\bar{\psi}\Sigma^{ab}\,, \tag{38}$$

where $\partial_c = e^\mu{}_c\partial_\mu$ and ω_{abc} are the spin connection coefficients, being antisymmetric in a, b due to vanishing non-metricity.

Computing the Hypermomentum

After some calculations, involving use of Cliffird algebra, we find (in agreement with [Hehl,Datta 1971])

$$\Delta^{\nu\mu\lambda} = -i\bar{\psi}\gamma^{[\lambda}\Sigma^{\mu\nu]}\psi = \frac{i}{2}\bar{\psi}\gamma^{[\nu}\gamma^{\mu}\gamma^{\lambda]}\psi \,. \qquad (39)$$

Improved tensor

Susbtituting this into our master eq. (22a) we find the improved (symmetric and conserved) EM tensor

$$T^{\mu\nu} = \frac{i}{4}\left(\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \partial^{\nu}\bar{\psi}\gamma^{\mu}\psi\right) + \frac{i}{4}\left(\bar{\psi}\gamma^{\nu}\partial^{\mu}\psi - \partial^{\mu}\bar{\psi}\gamma^{\nu}\psi\right) \,, \qquad (40)$$

In agreement with the standard textbook result!

A comment

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Why this works

With our method we can clearly see why in this case, this procedure works. The reason is that the hypermomentum is totally antisymmetric in this case. Then taking the symmetric part of (22a) the latter drops out and we get $T_{\mu\nu} = t_{(\mu\nu)}$.

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Can hypermomentum be used also for the improvement process of the e.m. tensor in a conformal field theory? An.: Yes!

Simplest example: massless, free scalar field

$$S_{\text{scalar}} = -\frac{1}{2}\int d^d x\,(\partial\phi)^2\,,\quad (\partial\phi)^2 = \eta^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi\,. \tag{41}$$

Canonical e.m. tensor is symmetric and coincides with the improved BR:

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial\phi)^2\,. \tag{42}$$

The same is also obtained by coupling to Einstein gravity. Action (41) is scale invariant → on-shell conserved dilatation current

$$j_D^\mu = x_\nu T^{\mu\nu} - V^\mu\,. \tag{43}$$

$$T^\mu{}_\mu = -\frac{d-2}{2}(\partial\phi)^2 = \partial_\mu V^\mu\,,\quad V^\mu = -\frac{d-2}{4}\partial^\mu\phi^2\,. \tag{44}$$

A traceless energy-momentum tensor can be obtained from the conformaly invariant action

$$S_{CCJ} = \int d^d x \sqrt{-g} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{8}\frac{d-2}{d-1}\tilde{R}\phi^2 \right) , \qquad (45)$$

Result:

$$\Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}\left(\eta_{\mu\nu}\partial^2\phi^2 - \partial_\mu\partial_\nu\phi^2\right) , \quad (46)$$

which is conserved, symmetric and also traceless

However...

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However...

1. For more complicated theories is not straightforward to come up with the required action.
2. The computation of $T_{\mu\nu}$ even though straightforward, is quite long.

It is remarkably simple to obtain the improved symmetric, conserved and now **traceless** e.m. tensor by coupling to Metric-Affine Geometry.

Just extend the derivative to

$$\partial_\mu \to \nabla_\mu \to D_\mu = \nabla_\mu + \frac{d-2}{d-1} S_\mu \,. \tag{47}$$

This then ensures that the action

$$S = -\frac{1}{2} \int d^d x \sqrt{-g}\, (D\phi)^2 \,, \tag{48}$$

is invariant under the local frame rescaling (see **D.I.**, S. Koivisto, 2019, Universe)

$$e_\mu{}^a \mapsto e^{\Omega} e_\mu{}^a \,, \quad \phi \mapsto e^{w\Omega}\phi \,, \quad w = \frac{d-2}{2} \,, \tag{49}$$

Before we plunge into calculations we already know that, by construction, the resulting canonical e.m. tensor will be traceless: $t = 0$.

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'Improved' to be Traceless

The improved traceless e.m. tensor is derived in (literally) one line. Compute hypermomentum and take the flat space limit:

$$\Delta_{\nu}{}^{\mu\lambda} = \frac{d-2}{d-1}\delta_{\nu}^{[\mu}\partial^{\lambda]}\phi^2 \,. \tag{50}$$

Then form (22a) we get

$$t^{\mu\nu} = \partial^{\mu}\phi\partial^{\nu}\phi - \frac{1}{2}\eta^{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}\left(\eta^{\mu\nu}\partial^2\phi^2 - \partial^{\mu}\partial^{\nu}\phi^2\right) \,, \tag{51}$$

which obviously coincides with (46) and is the symmetric, conserved and traceless e.m. tensor we were after!

Recap of the Algorithm

- **Step 1.** 'Go Metric-Affine', that is, promote $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ and $\partial_\mu \rightarrow \nabla_\mu$ where ∇_μ is a generic torsionful covariant derivative. Matter action $\Rightarrow S_M[g, \Gamma, \phi^A]$.

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- **Step 2.** Vary resulting action wrt affine connection and consequently go flat, i.e. set $g_{\mu\nu} \equiv \eta_{\mu\nu}$ and $\Gamma^\lambda{}_{\mu\nu} \equiv 0$ after the variation, viz.

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta \Gamma^\lambda{}_{\mu\nu}}\Big|_{g=\eta, \Gamma=0} \tag{52}$$

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- **Step 3.** Substitute the form of the hypermomentum found in the previous step into the flat space conservation law (22a) and solve for $T_{\mu\nu}$.
- **Outcome:** The resulting tensor $T_{\mu\nu}$ is the desired symmetric and conserved energy-momentum tensor.

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- Study non-trivial vacuua and the connection with Lorentz invariance breaking.
- Application of our result to non-Unitary CFTs.

...Thank you!!!