Multi-black holes at equilibrium in an external gravitational field

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(M.A. & A. Viganò)

- Binary and multi-black hole systems are acquiring more relevance in the understanding of the large scale structure and interactions of our universe, because gravitational waves observations. Anyway no exact analytical solution in GR are known.
- These systems generally are not isolated but they belong to larger gravitational structures such as galaxies, or clusters of galaxies which contribute to deform the gravitational field around the black holes and also the behaviour at large distances.
- ¿Is it possible to have multi black hole (at equilibrium) in pure GR? The double Kerr or the Israel-Khan solutions suffer from conical singularities which are not physical because are they are divergences in the spacetime, violate energy conditions and there are no observational nor experimental traces of their plausibility at the moment.
- Can the introduction of the external gravitational field help to fix these issues? Can we regularise these multi-black hole metrics from conical singularities and describe an realistic setting for binary black hole models? Surely we can, at least, circumvent the Israel theorem about black hole uniqueness.

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The multipolar gravitational background

The background can be written as a Weyl metric:

$$ds^2 = -e^{2\psi(\rho,z)}dt^2 + e^{-2\psi(\rho,z)}\left[e^{2\gamma(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] ,$$

with

$$\psi = \sum_{n=1}^{\infty}\left(\frac{a_n}{r^{n+1}} + b_n r^n\right)P_n ,$$

$$\gamma = \sum_{n,p=1}^{\infty}\left[\frac{(n+1)(p+1)a_n a_p}{(n+p+2)r^{n+p+2}}(P_{n+1}P_{p+1} - P_n P_p) + \frac{np b_n b_p r^{n+p}}{n+p}(P_n P_p - P_{n-1}P_{p-1})\right] ,$$

$r := \sqrt{\rho^2 + z^2}$ asymptotic radial coordinate and $P_n(z/r)$ Legendre polynomials
a_n describe the deformations of the source, b_n the external gravitational field whose distribution of matter located at large r.

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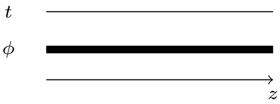
The internal and external multipole momenta $\mathcal{Q}_i^{\text{int}}$, $\mathcal{Q}_i^{\text{ext}}$

$$\mathcal{Q}_0^{\text{int}} = 0, \quad \mathcal{Q}_1^{\text{int}} = -a_1, \quad \mathcal{Q}_2^{\text{int}} = -a_2, \quad \mathcal{Q}_3^{\text{int}} = -a_3$$

$$\mathcal{Q}_0^{\text{ext}} = 0, \quad \mathcal{Q}_1^{\text{ext}} = -b_1, \quad \mathcal{Q}_2^{\text{ext}} = -b_2, \quad \mathcal{Q}_3^{\text{ext}} = \frac{b_1^3}{3} - b_3 . \tag{1}$$

Rod representation & solitons

Flat space $g_{ab} = \mathrm{diag}(-1, \rho^2)$



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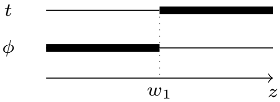

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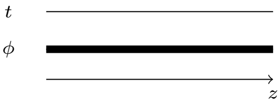

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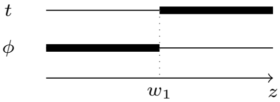
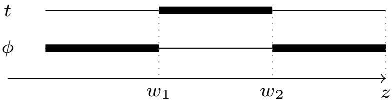


Schwazschild $g_{ab} = \mathrm{diag}\left(-\frac{\mu_1}{\mu_2}, \rho^2 \frac{\mu_2}{\mu_1}\right)$

$\mu_k(\rho, z) = w_k - z + \sqrt{\rho^2 + (z - w_k)^2}$



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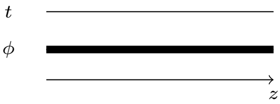

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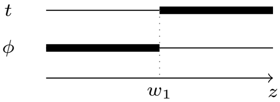

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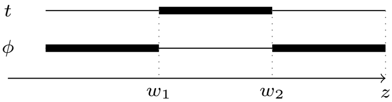
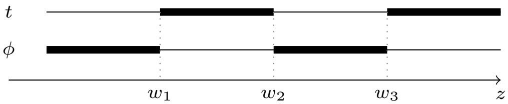


C-metric $g_{ab} = \mathrm{diag}\left(-\frac{\mu_1\mu_3}{\mu_2}, \frac{\rho^2\mu_2}{\mu_1\mu_3}\right)$



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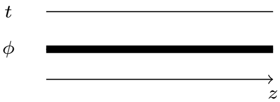

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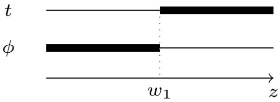

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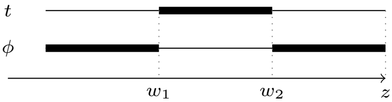

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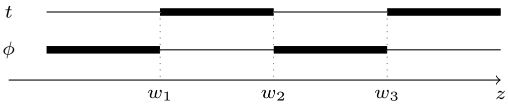
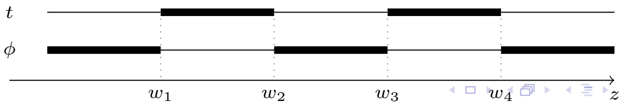
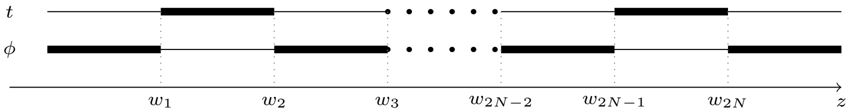


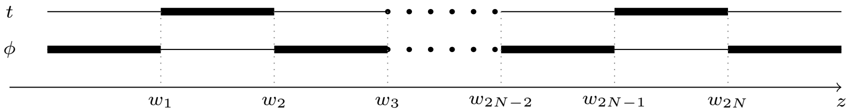
Binary Black Hole $g_{ab} = \mathrm{diag}\left(-\frac{\mu_1\mu_3}{\mu_2\mu_4}, \rho^2 \frac{\mu_2\mu_4}{\mu_1\mu_3}\right)$



Array of N black holes in an external gravitational field



Each couple of solitons adds a black hole, then the addition of $2N$ solitons gives rise to a spacetime containing N black holes, whose metric is



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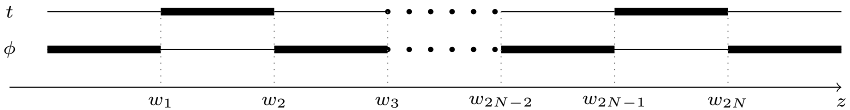
$$ds^2 = f(\rho, z)(d\rho^2 + dz^2) + g_{ab}(\rho, z)dx^a dx^b \ ,$$

Array of N black holes in an external gravitational field



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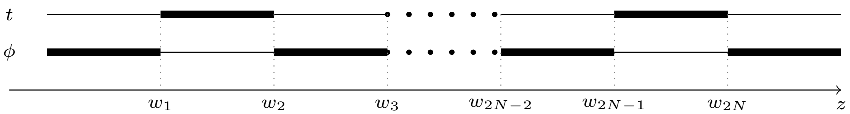
$$f = 16 C_f f_0 \left(\prod_{k=1}^{N} \mu_{2k}^{2N+1} \mu_{2k-1}^{2N-1} \right) \left(\prod_{k=1}^{2N} \frac{1}{\rho^2 + \mu_k^2} \right) \left(\prod_{k=1, l=1,3,\cdots}^{2N-1} \frac{1}{(\mu_k - \mu_{k+l})^2} \right)$$

$$\times \left(\prod_{k=1, l=2,4,\cdots}^{2N-2} \frac{1}{(\rho^2 + \mu_k \mu_{k+l})^2} \right) \exp\left[2 \sum_{k=1}^{2N} (-1)^{k+1} F(\rho, z, \mu_k) \right] .$$

Array of N black holes in an external gravitational field



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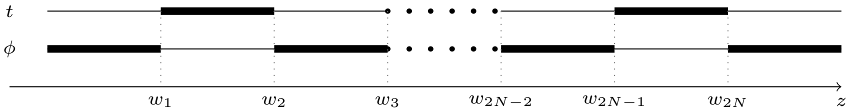
$$F(\rho, z, \lambda) = 2 \sum_{n=1}^{\infty} b_n \left[\sum_{l=0}^{\infty} \binom{n}{l} \left(\frac{-\rho^2}{2\lambda} \right)^l \left(z + \frac{\lambda}{2} \right)^{n-l} - \sum_{l=1}^{n} \sum_{k=0}^{[(n-l)/2]} \frac{(-1)^{k+l} 2^{-2k-l} n! \lambda^{-l}}{k!(k+l)!(n-2k-l)!} \frac{\rho^{2(k+l)}}{z^{2k+l-n}} \right]$$

Array of N black holes in an external gravitational field



Each couple of solitons adds a black hole, then the addition of $2N$ solitons gives rise to a spacetime containing N static black holes, whose metric is

$$ds^2 = f(\rho,z)(d\rho^2 + dz^2) + g_{ab}(\rho,z)dx^a dx^b \,,$$

$$g_{ab} = \mathrm{diag}\left[-\frac{\prod_{k=1}^{N} \mu_{2k-1}}{\prod_{l=1}^{N} \mu_{2l}} \exp\left(2\sum_{n=1}^{\infty} b_n r^n P_n\right), \rho^2 \frac{\prod_{l=1}^{N} \mu_{2l}}{\prod_{k=1}^{N} \mu_{2k-1}} \exp\left(-2\sum_{n=1}^{\infty} b_n r^n P_n\right)\right],$$

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The 2N parameters are related to the mass m_i and position z_i of the N black hole:

$$w_1 = z_1 - m_1 \,, \quad w_2 = z_1 + m_1 \,, \quad \dots \quad w_{2N-1} = z_N - m_N \,, \quad w_{2N} = z_N + m_N \,.$$

Conical singularities and regularisation

Multipole momenta b_n allow to remove the conical singularities from the metric requiring $\mathcal{P} \equiv f g_{tt} \to 1$ as $\rho \to 0$ between the k-th and $(k+1)$-th rod $(w_{2k} < z < w_{2k+1})$, for $1 \leq k < N$.

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$$C_f = 2^{2(2N+1)} \left[\prod_{i=1}^{N} (w_{2i} - w_{2i-1})^2 \right] \left[\prod_{k=1}^{N-1} \prod_{j=1}^{N-k} (w_{2k-1} - w_{2k+2j})^2 (w_{2k} - w_{2k+2j-1})^2 \right] .$$

$$\mathcal{P}_k = f g_{tt} = \left[\prod_{i=1}^{2k} \prod_{j=2k+1}^{2N} (w_j - w_i)^{2\,(-1)^{i+j+1}} \right] \exp \left[4 \sum_{n=1}^{\infty} b_n \sum_{j=2k+1}^{2N} (-1)^{j+1} w_j^n \right] ,$$

while in the region $z > w_{2N}$ $P_N = 1$ and for $z < w_1$

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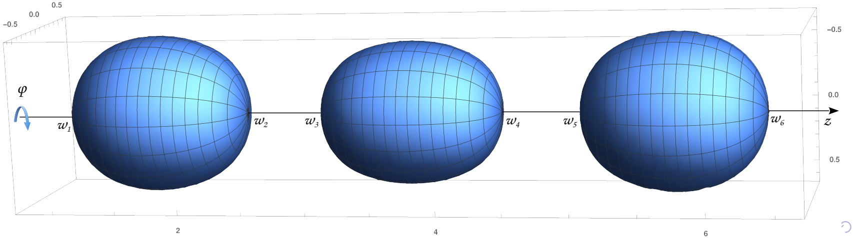

$N = 1$ Distorted Schwarzschild black hole

For $N = 1$ static black hole embedded in an external gravitational field

$$e^{2\psi} = -\frac{\mu_1}{\mu_2} \exp\left(2 \sum_{n=1}^{\infty} b_n r^n P_n \right) ,$$

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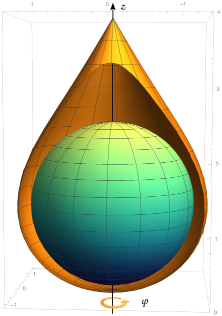
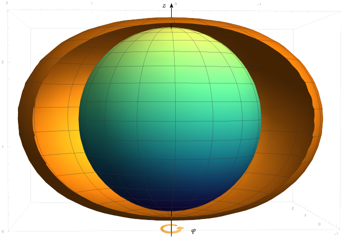




$N = 1$ Distorted Schwarzschild black hole

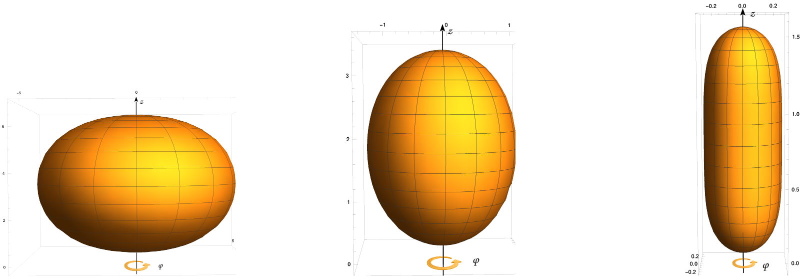
For $N = 1$ static black hole embedded in an external gravitational field

$$e^{2\psi} = -\frac{\mu_1}{\mu_2} \exp\left(2 \sum_{n=1}^{\infty} b_n r^n P_n \right) ,$$

$$e^{2\gamma} = \frac{16\ C_f\ f_0\ \mu_2^3\ \mu_1\ e^{2F(\rho,z,\mu_1)-2F(\rho,z,\mu_2)}}{(\rho^2+\mu_1^2)(\rho^2+\mu_2^2)(\mu_1-\mu_2)^2} .$$

Schwarzschild limit for $b_n = 0 \quad \forall n$.

$$\rho = \sqrt{r(r-2m)}\sin\theta , \qquad z = z_1 + (r-m)\cos\theta .$$



(a) $m = 0.5,\ b_2 = 0.4,\ z_1 = 2$ (b) $m = 1,\ b_2 = -0.1,\ z_1 = 2$ (c) $m = 1,\ b_2 = -0.5,\ z_1 = 2$

$N = 2$ Generating a Charged Binary Black Hole system

Seed: $N = 2$, multipolar generalisation of the Bach-Weyl solution

$$e^{2\psi} = \frac{\mu_1 \mu_3}{\mu_2 \mu_4} \exp\left[2b_1 z + 2b_2 \left(z^2 - \frac{\rho^2}{2}\right)\right],$$

$$e^{2\gamma} = \frac{16 C_f \, e^{2\psi} \, \mu_1^3 \mu_2^5 \mu_3^3 \mu_4^5}{W_{11} W_{22} W_{33} W_{44} W_{13}^2 W_{24}^2 Y_{12} Y_{14} Y_{23} Y_{34}} \exp\Big\{ -b_1^2 \rho^2 + \frac{b_2^2}{2}\left(\rho^2 - 8z^2\right)\rho^2 - 4b_1 b_2 z \rho^2$$
$$+ 2b_1(-z + \mu_1 - \mu_2 + \mu_3 - \mu_4) + b_2\left[-2z^2 + \rho^2 + 4z(\mu_1 - \mu_2) + \mu_1^2 - \mu_2^2\right.$$
$$\left. + (\mu_3 - \mu_4)(4z + \mu_3 + \mu_4)\right] \Big\},$$

where $W_{ij} = \rho^2 + \mu_i \mu_j$, $Y_{ij} = (\mu_i - \mu_j)^2$ and $\mu_k(\rho, z) = w_k - z + \sqrt{\rho^2 + (z - w_k)^2}$

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Charging transformation

$$e^{2\psi} \to e^{2\hat{\psi}} = \frac{e^{2\psi}(1 - \zeta^2)^2}{(1 - \zeta^2 e^{2\psi})^2},$$

which is be supported by an electric field given by

$$\hat{A}_\mu = \left(\frac{\zeta(e^{2\psi} - 1)}{1 - \zeta^2 e^{2\psi}}, 0, 0, 0\right).$$

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The usual inverse scattering parametrisation for the black holes is used:

$$w_1 = z_1 - \sigma_1\,, \quad w_2 = z_1 + \sigma_1\,, \quad w_3 = z_2 - \sigma_2\,, \quad w_4 = z_2 + \sigma_2\,,$$

Regularising condition $f g_{tt} = 1$
for $\rho = 0$ and $z \in (-\infty, w_1)$, $z \in (w_2, w_3)$, $z \in (w_4, \infty)$

$$
\begin{aligned}
C_f &= 16(w_1 - w_2)^2 (w_2 - w_3)^2 (w_1 - w_4)^2 (w_3 - w_4)^2 \,, \\
b_1 &= \frac{w_1^2 - w_2^2 + w_3^2 - w_4^2}{2(w_1 - w_2)(w_1 + w_2 - w_3 - w_4)(w_3 - w_4)} \log\left[\frac{(w_1 - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)}\right] , \\
b_2 &= -\frac{w_1 - w_2 + w_3 - w_4}{2(w_1 - w_2)(w_1 + w_2 - w_3 - w_4)(w_3 - w_4)} \log\left[\frac{(w_1 - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)}\right] .
\end{aligned}
$$

$N = 2$ Regularising the Charged Binary Black Hole system

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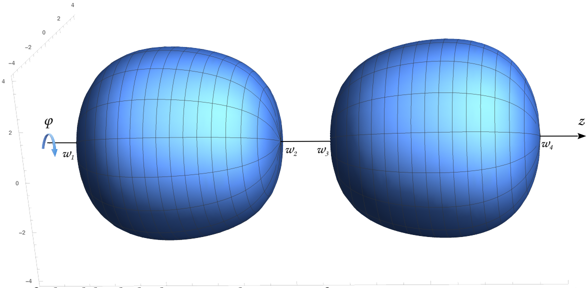
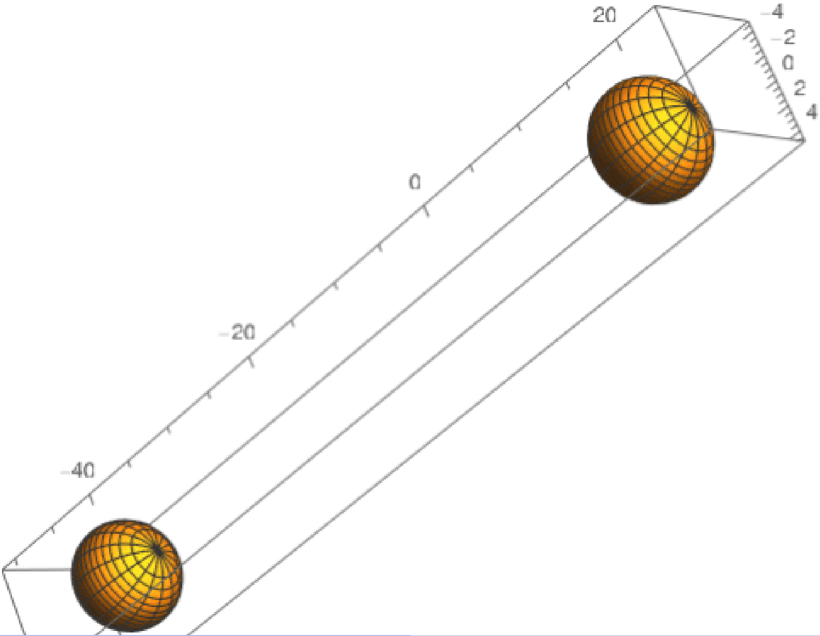
\end{aligned}
$$



$\mathbb{E}^3$ Embedding of the black hole event horizons for $z_1 = 5$, $z_2 = 15$, $m_1 = 4$, $m_2 = 4$.

Twin binary BH system for different separation $z_1 - z_2$



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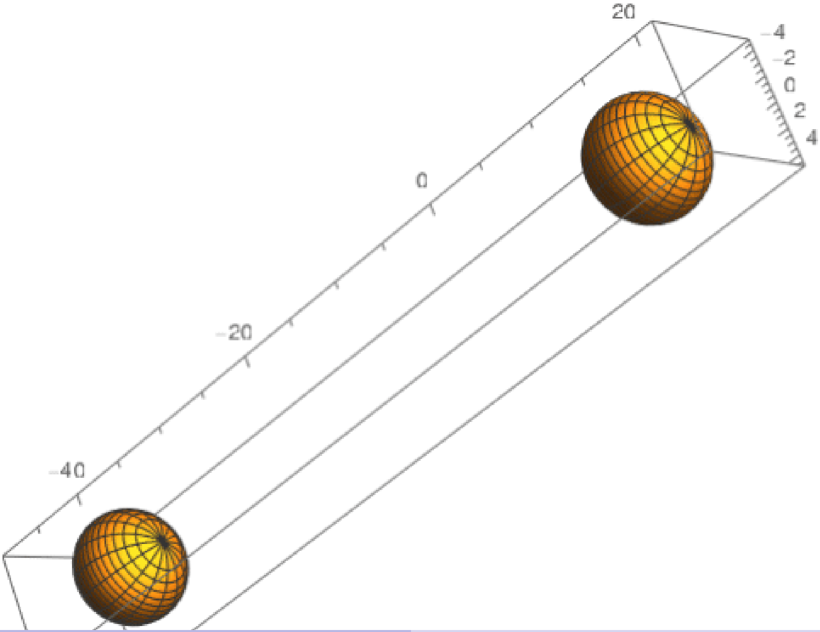

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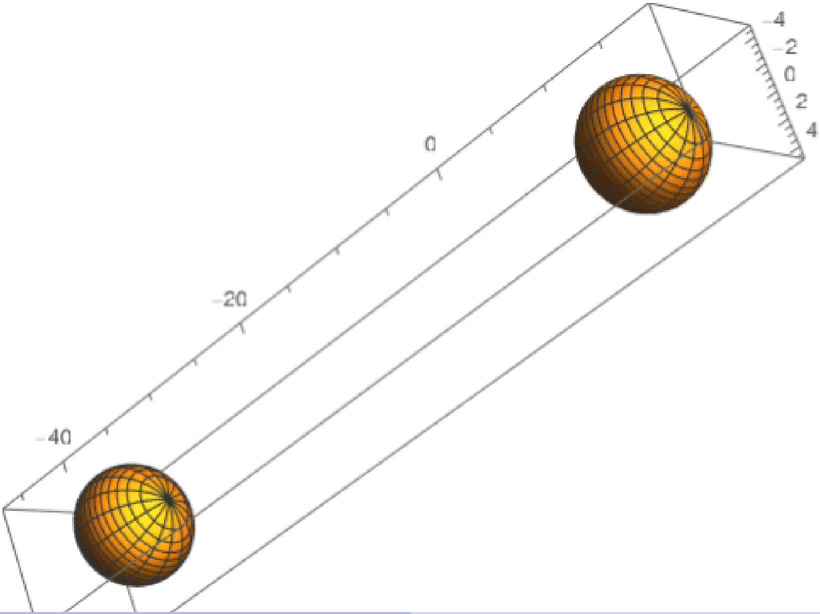

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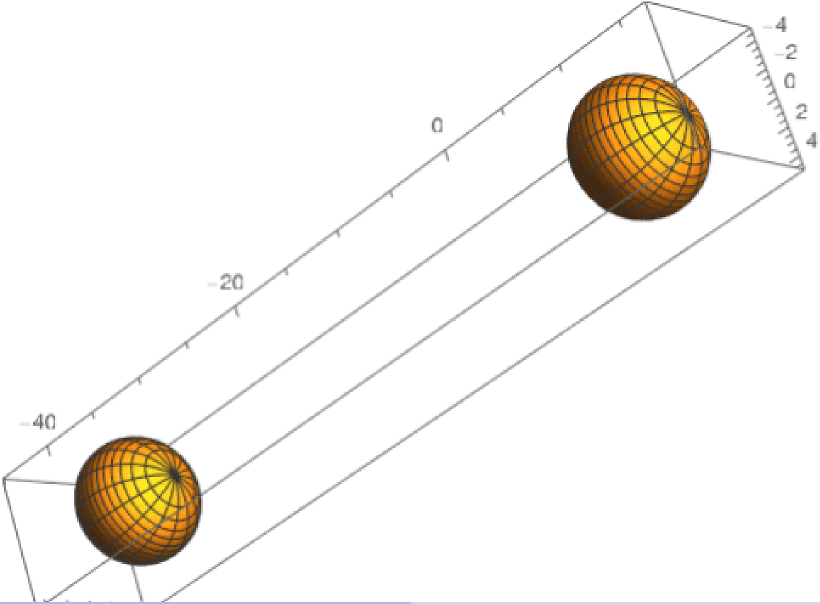

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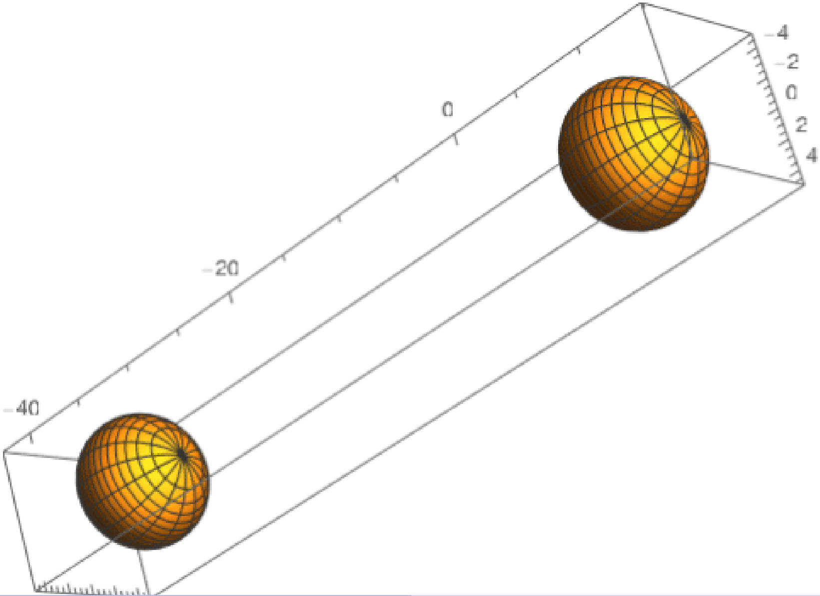

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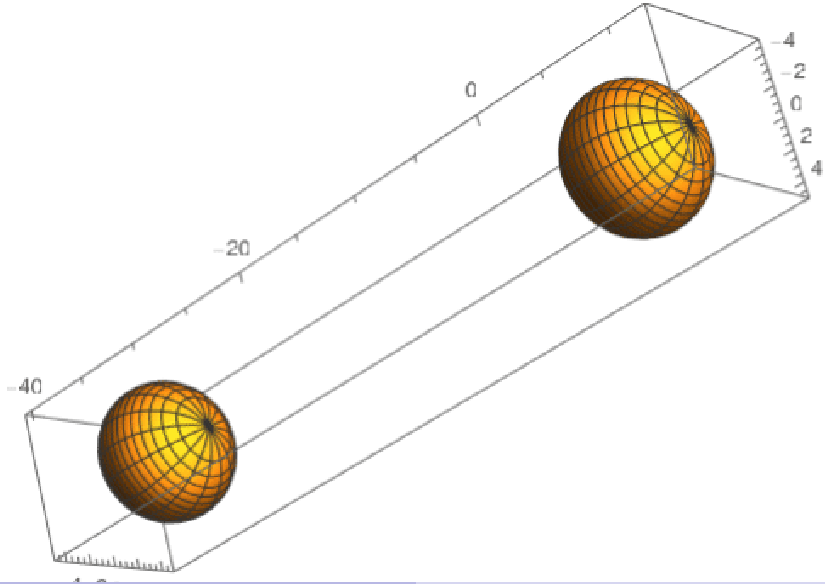

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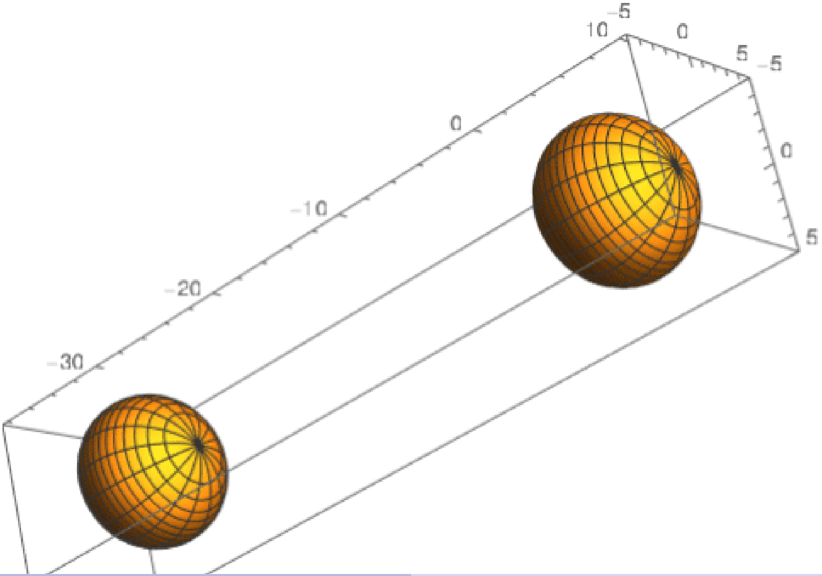
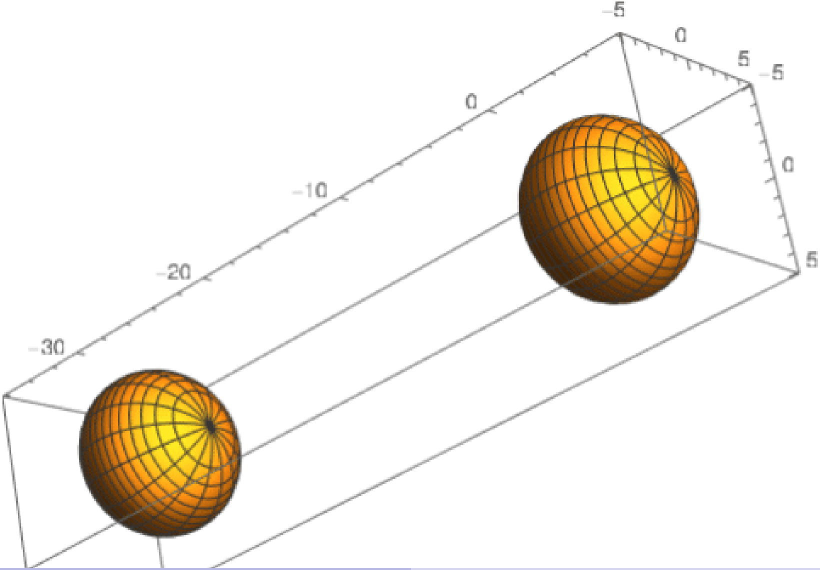
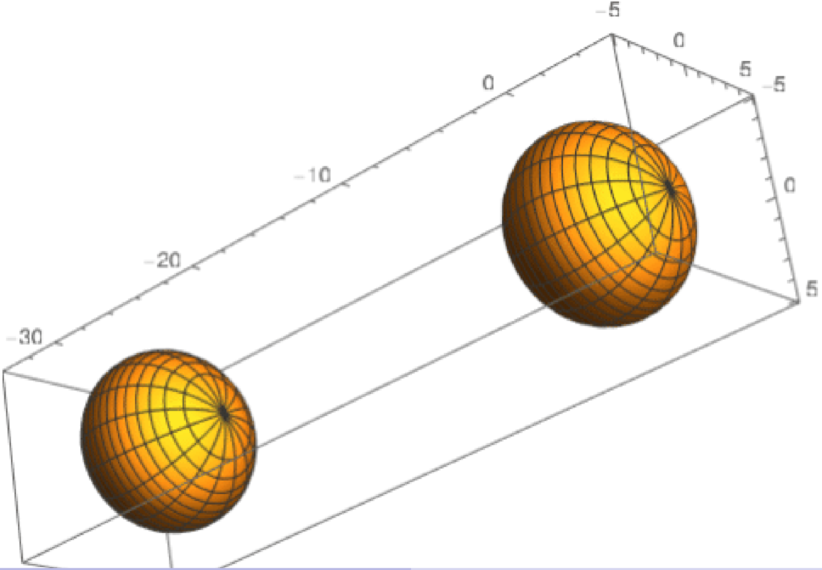



-5
0
5
-5
0
-10
-20
-30
0
5

Twin binary BH system for different separation $z_1 - z_2$



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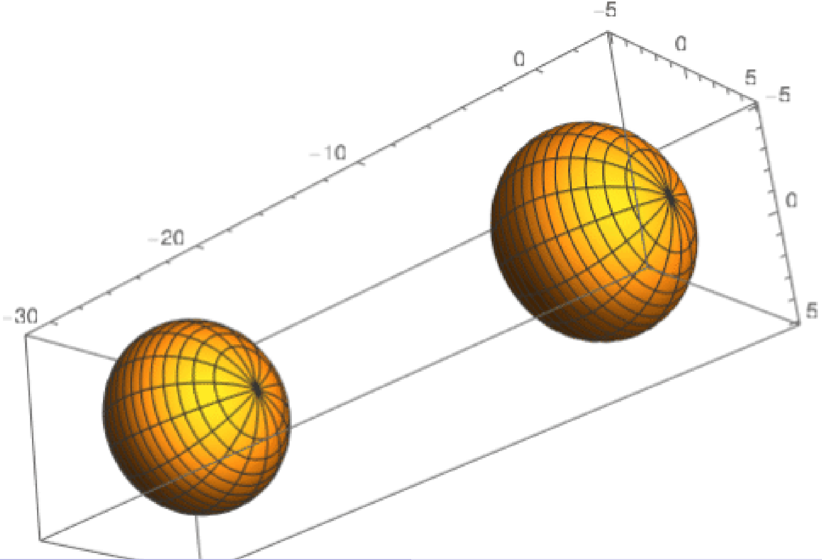

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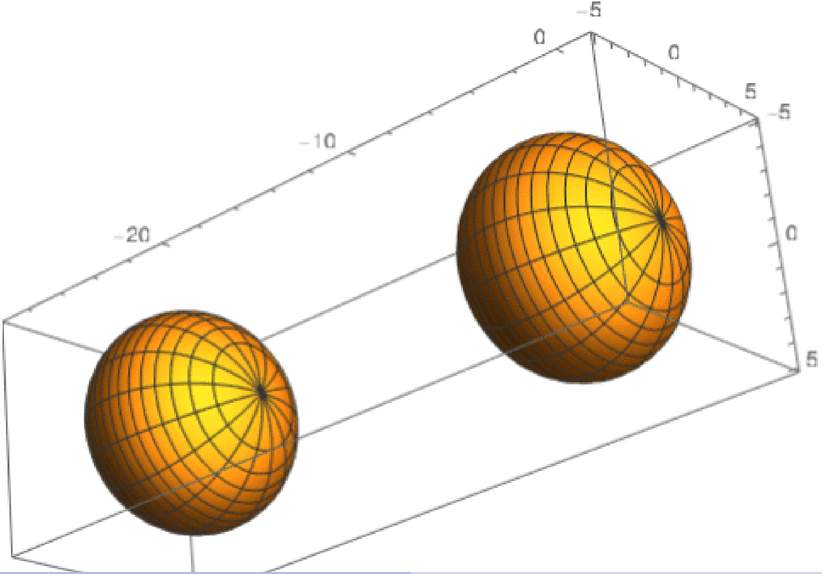
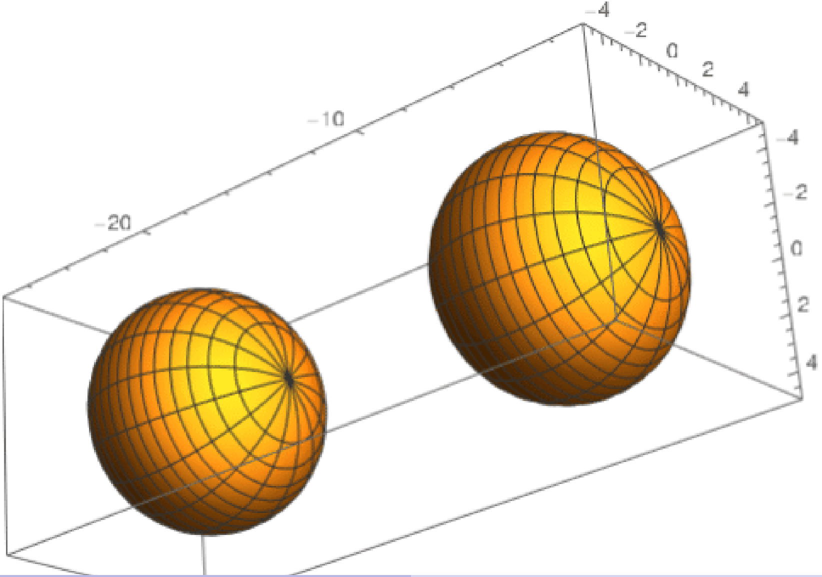
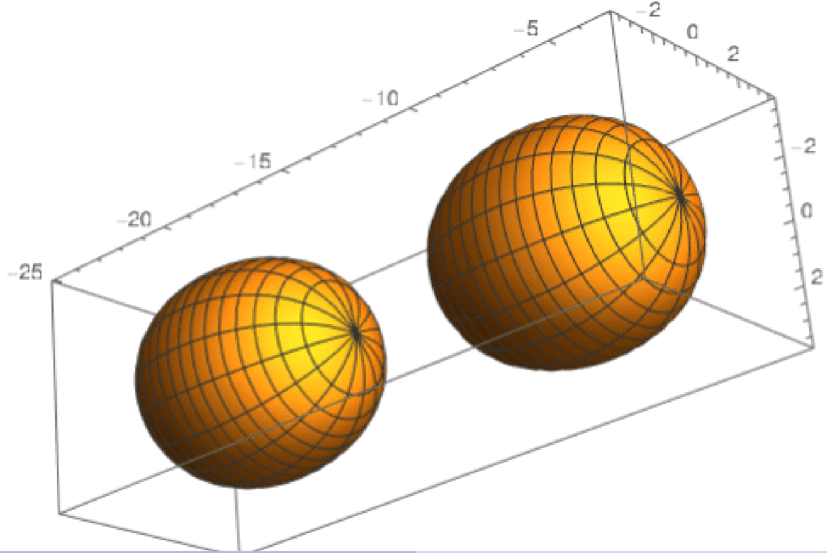



-4
-2
0
2
4
-10
-20
-4
-2
0
2
4


-25
-20
-15
-10
-5
-2
0
2
-2
0
2

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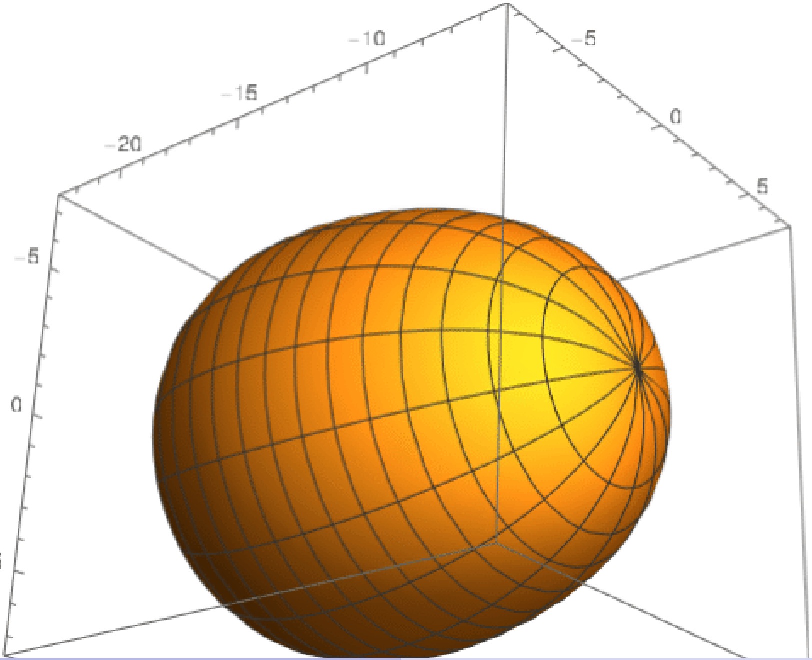

$N = 2$: Conserved Charges of the Black Hole Binary

The electric charge

$$Q_i = -\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{w_{2i-1}}^{w_{2i}} dz\, \rho\, g_{tt}^{-1} \partial_\rho A_t\big|_{\rho=0} = \frac{2\zeta\sigma_i}{1-\zeta^2}\,.$$

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Masses anche electric charges are not 4 independent quantities but just three

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$$S_2 = \pi \frac{(w_4-w_3)^2(w_4-w_1)}{(w_4-w_2)(1-\zeta^2)^2} e^{-2w_4(b_1+b_2 w_4)}\,.$$

Smarr law for the charged binary system

Temperature of the event horizons

$$T_i = \frac{\alpha \sigma_i}{2 S_i} \; .$$

where

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The Coulomb electric potential Θ evaluated on both the event horizons takes the same value

$$\Theta = -\xi^\mu A_\mu \big|_{\rho=0} = \alpha(\zeta - \Theta_\infty) \, .$$

Smarr law

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Note that the proper distance between the two event horizon surfaces is finite:

$$\ell = \int_{w_2}^{w_3} \sqrt{g_{zz}} dz < \infty \, .$$

Majumdar–Papapetrou limit

The regularising constraints have some special points for which $b_i = 0 \ \forall i$:

$$\{w_1 = w_2 \quad , \quad w_3 = w_4\} \quad \Longrightarrow z_i - \sigma_i = z_i + \sigma_i \quad \Longrightarrow \sigma_i = \sqrt{M_i^2 - Q_i^2} = 0$$

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$$d\hat{s}^2 = -e^{2\hat{\psi}} dt^2 + e^{-2\hat{\psi}} \left(d\rho^2 + dz^2 + \rho^2 d\phi \right) ,$$

$$\hat{A}_t = \left(1 + \frac{M_1}{\sqrt{\rho^2 + (z - z_1)^2}} + \frac{M_2}{\sqrt{\rho^2 + (z - z_2)^2}} \right)^{-1} ,$$

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which is exactly the double Majumdar–Papapetrou metric.

Summary, Conclusions & Pespectives

- Analytical multi black hole solutions at equilibrium can be build in General Relativity without conical singularities. The equilibrium is provided by a generic multipolar external gravitational field.
- We show how to build charged, rotating, NUTty, accelerating extensions, thanks to solution generating techniques, providing the most general black hole family of pure GR. Smarr, first and second law can be explicitly verified.
  These metrics, in certain limits, can also describe a large family of particle in GR.
- The External field can also provide a motivation for accelerating black holes and C-metric opening the possibility of a pair production of black holes in external gravitational field.
- Generalisation to other gravitational backgrounds or to other gravitational theories such as scalar tensor theories, Brans-Dicke or $f(R)$ can be pursuit.

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